TO

Alan Lord *Brodrick*, Baro
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p.p.

A

TREATISE

OF

ALGEBRA

In Two BOOKS:

The First treating of the *Arithmetical*, and the Second of the *Geometrical* Part.

The SECOND EDITION with ADDITIONS.

By PHILIP RONAYNE, Gent.

LONDON:

Printed for WILLIAM and JOHN INNYS, at the West End of St. *Paul's*
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TO

His Excellency *Alan* Lord *Brodrick*, Baron of *Middleton*, one of the Lords Justices, and Lord High-Chancellor of the Kingdom of *Ireland*.

May it please Your Excellency,

THE following *Treatiſe* might juſtly be intitled to your Patronage, were the Performance ſuitable to the Dignity of the Subject, which is a Science, that exceeds all others in Certainty and Clearneſs, and, by a Natural and Regular Method, conduces much to the opening and enlarging of the Mind.

BUT, tho' this ſmall Volume can't pretend to any ſuch Perfection, yet, as it may, in ſome Meaſure, advance ſo uſeful a part of Learning, I take the Liberty to put it under your Protection, *My Lord*, who gave early Proofs of a clear Underſtanding, an exact Judgment, and a vaſt Genius capable of comprehending and improving the moſt abſtruſe Sciences.

As these Qualities made you eminent in your Profession, so your Integrity gain'd you such an Universal Esteem, that, with the general Approbation of the People, you were soon called into several great Stations, and, at length, deservedly advanced to the highest Dignities in this Kingdom, the Publick Welfare whereof you have constantly promoted in all the Employments and Offices, which you have so honourably filled; And, that you may long live the great Ornament thereof, is the hearty Wish of,

My Lord,

Your Excellency's

Most Humble, and

Most Obedient Servant

CORKE, *August* the 2d 1717.

PHILIP RONAYNE.

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INTRO-

INTRODUCTION.

MY Deſign by this *Introduction*, is to put my Reader in Mind of ſome Things in Arithmetick, which are not commonly taken Notice of in Books on that Subject; *viz.* that,

I. There have been ſome Properties attributed to Numbers, which are only the Reſult of the Method of Notation now in uſe: As for Inſtance,

'Tis commonly ſaid and allowed for Truth in the known common Way of proving Multiplication and Diviſion by caſting away the Nines, that

Propoſition.

The Number 9 has that Property, that any Number whatſoever Divided by it, ſhall leave the ſame Remainder, as if the Sum of the Figures compoſing the ſaid Number were Divided by 9.

I ſay, this is true in the common Method of Notation: But it is to the ſaid Method only, that this Fact is owing, as will appear from the following Demonſtration of the Propoſition above recited, and the Scholium annexed, the former of which I had from the Reverend Mr. *Chinery* of *Middleton*, near *Corke*, in the following Expreſſions.

For the clearer Demonſtration of this Propoſition, I ſhall premiſe two ſelf evident Lemma's.

Lemma 1.

The Local Value of any Figure is equal to the Rectangle of its Simple Value, and the Denomination of its place: Thus $6^{*} = 6 \times 1$, $60 = 6 \times 10$, $600 = 6 \times 100$, *&c.*

Lemma 2.

To Multiply or Divide one Number by an other, is in Conclusion the same Thing with Multiplying or Dividing respectively the Sum of the Parts of the former by the latter: Thus, because $10 = 9 + 1$; therefore $6 \times 10 = 6 \times 9 + 6 \times 1$; and, because $100 = 99 + 1$; therefore $6 \times 100 = 6 \times 99 + 6 \times 1$; *&c.* And so $\frac{10}{6} = \frac{9}{6} + \frac{1}{6}$, and $\frac{100}{6} = \frac{99}{6} + \frac{1}{6}$, *&c.*

These Lemma's premis'd, I proceed to the Demonstration of the Propos'd Proposition.

Demonstration.

Let the given Number be $3467 = \left\{\begin{array}{l} 7 \times 1 \\ + 6 \times 10 \\ + 4 \times 100 \\ + 3 \times 1000 \end{array}\right\}$ per *Lemma* 1. $= \left\{\begin{array}{r} 7 \times 1 \\ + 6 \times 9 + 6 \times 1 \\ + 4 \times 99 + 4 \times 1 \\ + 3 \times 999 + 3 \times 1 \end{array}\right\}$ per *Lemma* 2.; and consequently $\frac{3467}{9} = \frac{6 \times 9 + 4 \times 99 + 3 \times 999}{9} + \frac{7 \times 1 + 6 \times 1 + 4 \times 1 + 3 \times 1}{9}$ (per *Lemma* 2.); but 6×9, 4×99, 3×999 being Multiples of 9, their Sum must also be a Multiple thereof, and consequently Divided by it will leave no Remainder; therefore the Remainder of 3467 Divided by 9 will be equal to the Remainder of $7 \times 1 + 6 \times 1 + 4 \times 1 + 3 \times 1$ Divided by 9; *i. e.* equal to the Remainder of $7 + 6 + 4 + 3$ Divided by 9. The like Demonstration may evidently be applyed as well to any other Number as to 3467. *Q. E. D.*

* *See the following Notation.*

Scholium

Scholium.

From the foregoing Demonſtration it is evident that any Number deſign'd by any Method of Notation after the common one, being Divided by : 10 — 1 : ſhall leave the ſame Remainder with that which wou'd remain if the Sum of the Figures compoſing that Number were likewiſe Divided by : 10 — 1 :

Thus, ſuppoſing the Method of Notation, I wou'd uſe deſignable by only the five firſt Figures of them which are commonly us'd; *viz.* that 0, 1, 2, 3, 4 ſhall deſign, as uſual, nothing, one, two, three, four; but 10, 11, 12, 13, 14 ſhall deſign five, ſix, ſeven, eight, nine; and 20, 21, 22, 23; *&c.* ſhall deſign ten, eleven, twelve, thirteen; *&c.*

And Suppoſe 242 (or twice twenty five more four times five more two, *viz.* ſeventy two) be propos'd to be Divided by : 10 — 1 : which, according to this Method of Notation, is = 4 :

Then 4) 242 (33 = three times five more three = eighteen
22
——
0 Remainder.

is the Quotient : But the Remainder is 0 : And 2 + 4 + 2 (or eight) = 13 being Divided by 4 is = 2 exactly without any Remainder ; Or 422 (or one hundred and twelve) ÷ 4 = 103 or twenty eight juſt. *&c.*

II. In ſome Operations in Multiplication and Diviſion, it will be requiſite to take Notice of the following Definitions; *viz.*

That every Multiplication is a Proportion in which 1 (or Unity) is to the Multiplicator, as the Multiplicand is to the Product ſought.

Likewiſe that every Diviſion is a Proportion in which the Diviſor is to the Dividend, as 1 is to the Quotient ſought :

So ; if it were required to Multiply 4 *s.* 6 *d.* by 4 *s.* 6 *d.*

Firſt, Reducing it to a Proportion, 1 .. 4 *s.* 6 *d.* : : 4 *s.* 6 *d.* .. the fourth Number ſought.

Now, it being uncertain what Name the firſt Term 1 is off (*viz.* whether it be a Pound, Shilling, or a Penny) 'tis therefore uncertain what the Product or fourth Number ſought is : Thus, if the firſt Term 1 be 1 *l.* then the Product is $\frac{2916}{240}$ *d.* = 12 $\frac{36}{240}$ *d.* If the ſaid Term 1 be 1 *s.* then the Product is $\frac{2916}{12}$ *d.* = 243 *d.* = 20$\frac{1}{4}$ *s.* And, if the ſaid Term be 1 *d.* then the Product, or fourth Number ſought, in the foregoing Proportion, is 2916 *d.* = 243 *s.* = 12 *l.* 3 *s.* 00 *d.*

The like is to be underſtood of Diviſion.

III. Tho' the Rule of Double Position will not extend to Questions producing Quadratick or Superior Equations, nor even to some simple ones; yet, in Cases of Approximations, it may be us'd to good Purpose in some Abstruse Questions. By way of Example take the following Question and Solution.

Two Men D and E made a Stock of 165 *l.* D's Money was in Company for 3 Months, and E's was in for 2 Months. When they shar'd Stock and Gain, D receiv'd for his Share 40 *l.* and E 153 *l.* I demand D's Stock?

By the *Theorem* in *Question* 3. Chap. 2. Part XI. of the 1st Book of this Treatise, you'll find D's Stock $= \sqrt{48169} - 187 = 32.47437208 + l.$

But my present Design is to shew how to approach to this Stock of D's by the Help of Double Position. Thus,

First $40 + 153 = 193$, and $193 - 165 = 28 l.$ = their whole Gain.

Next, Supposing first D's Stock = 20 *l.* I find it by much too little; again, Supposing it = 30 *l.* I find it still too little; but supposing it = 33 *l.* I find it somewhat too much; therefore I suppose it = 32 *l.* then E's Stock will be 133 *l.*

$$\begin{array}{r} 32 \\ 3 \\ \hline 96 \end{array} \qquad \begin{array}{r} 133 \\ 2 \\ \hline 266 \\ 96 \\ \hline 362 \end{array}$$

$$362 .. 28 l. :: 96 \quad 7.4254 + l.$$

$$\begin{array}{r} 32 \\ \hline 40 - 39.4254 = .5745 \text{ Error.} \end{array}$$

Suppose D's Stock to be 32.5 *l.* then E's Stock will be 132.5 *l.*

$$\begin{array}{r} 32.5 \\ 3 \\ \hline 97.5 \end{array} \qquad \begin{array}{r} 132.5 \\ 2 \\ \hline 265.0 \\ 97.5 \\ \hline 362.5 \end{array}$$

$$362.5 .. 28 l. :: 97.5 .. 7.5310 + l.$$

$$\begin{array}{r} 32.5 \\ \hline 40.0310 \\ 40 \\ \hline .0310 \text{ Error.} \end{array}$$

32	32.5
.5745	.0310
18.67125	.9920
.9920	
19.66325	

.5745
.0310
.6055 } for the Errors are unlike.

And $19.66325 \div .6055 = 32.4744\,l. = 32\,l.\ 09\,s.\ 5\tfrac{3}{4}\,d.$ + = D's Stock Anſwer.

But if this Anſwer be not thought near enough the Truth; Suppoſe, again D's Stock $= 32.4744\,l.$ and, then find the Error by the foregoing Method, and it will be found to conſiſt in Exceſs: And, for a ſecond Suppoſition, put D's Stock $= 32.4743\,l.$ and proceed, in the foregoing Manner, in finding the Error, which will be found to conſiſt in Defect. Then by theſe Poſitions and Errors (the Errors being purſued to a good many places in Decimals) the Anſwer will be found very near the Truth, only purſuing this one Operation by Double Poſition.

ERRATA.

ERRATA.

IN the running Title in Pages 2, 4 and 6, read Of Quantities; as also in *p.* 3 and 5. *r.* Chap. I. Notation, and in *p.* 8, *r.* Of whole Quantities.

Page 6, Line 1, for 6, *r.* 9. *p.* 32, *l.* 12, after $\frac{ab}{c}$ place .; and instead of t put T. *p.* 49, *l.* the last, for $+$, *r.* $\pm$. *p.* 82, *l.* the last, for Single, *r.* Simple. *p.* 117, *l.* 22, for of Notice, *r.* Notice of. *p.* 135, *l.* the last but ten, for join, *r.* join'd. *p.* 146, last *l.* but three, for 11, *r.* 12. *p.* 153, *l.* the last but three, for fought, *r.* sought. *p.* 153, *l.* 26 and 27, for abded, *r.* added. *p.* 159, *l.* 12, *r.* $\overline{\frac{ae}{a+e}}\Big|^2$.

Page 179, *l.* the last but four, for $-216d$, *r.* $-216d^3$. *Note*, Dele the 11th and 12th *l.* (*viz.* the *N. B.*) in *p.* 180: And, instead of the 7 last *l.* in *p.* 191, and all in *p.* 192, *r.*

1. If the Terms of the Equation be not in the least Integers, or, if any of them be Fractions, Reduce them to the least Integers, by Multiplication, or Division.

2. Next, reduce all the Terms of this Equation to one Side thereof, and the other Side will be 0.

3. Then, find all the Divisors of the absolute Number in this Equation; as also all the Divisors of the Co-efficient of the greatest Power of the Root sought therein (by the foregoing *Lemma*); and try whether any of the former Divisors, connected by $-$ or $+$ to any of the latter Multiplyed by the unknown Root *a*, will Divide the total Sum of the last mention'd Equation, without leaving a Remainder. For, when such Division succeeds, either the known Part of the Residual or Binomial Divisor, with a contrary Sign, Divided by the Co-efficient of *a* therein, is the desir'd Value of *a*, or, at least, the Quotients give an Equation which hath fewer Dimensions by one than the Equation Divided: And if this Equation contains three or more Dimensions, let it be examin'd by Division, as before; and so on. By which Divisions the Roots of the propos'd Equation may sometimes be made known, or the Equation be reduc'd to a Quadratick one; and then the sought Root will be found by the *Canons* given for Solving Adfected Quadraticks.

Ex-

ERRATA.

Examples.

1. If $a^3 - \frac{1}{6}aa + 2a = \frac{1}{3}$; Quere the Values, or Value of a?

In order to reduce this Equation to the leaſt Integer Terms, I Multiply it by 6, and the Products are $6a^3 - aa + 12a = 2$: And, by tranſpoſing 2, we have $6a^3 - aa + 12a - 2 = 0$.

The Diviſors of the abſolute Number 2 are 1 and 2; and the Diviſors of 6, the Co-efficient of the greateſt Power of the Root in the Equation, are 1, 2, 3 and 6: Wherefore I try whether $a - 1$ or $a + 1$, $a - 2$ or $a + 2$; $2a - 1$ or $2a + 1$; $3a - 1$ or $3a + 1$, $3a - 2$ or $3a + 2$ will Divide the Equation without leaving a Remainder (for, ſeeing $a - 1$ or $a + 1$ will not ſucceed, I can't expect that any of its Multiples will; and therefore I paſs by $2a - 2$, $2a + 2$; $3a - 3$ and $3a + 3$); but, finding neither of them will do, I try next with $6a - 1$, which will exactly do; and therefore a is $= 1 \div 6 = \frac{1}{6}$: And the Quotients being $aa + 2 = 0$, the other two Roots or Values of a are Imaginary; wherefore the only real Value of a in the above Equation is 1.

This Method being only Tentative, and, in ſome Caſes, exceeding Tedious, I will therefore only add one Example more, in which a convenient Contraction is us'd; from which Ev: Sir Iſaac Newton's *Method of finding compound rational Diviſors of one Dimenſion,* &c. *may be deduc'd and Demonſtrated.*

2. Let it be required to find the Roots of this Equation $a^3 - 36aa + 2a - 72 = 0$.

The abſolute Number 72 can be exactly Divided by 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72: And 1 being the Co-efficient of the greateſt Power of a in the Equation, its Diviſor is only 1; wherefore the propos'd Equation is to be Divided by $a -$ and $+ 1$, $a -$ and $+ 2$, *&c.* to find ſuch of them as will exactly do it; but, ſince here are a great many Diviſors, and that (by the Compoſition of Equations) there can be, at moſt, but three ſuch Diviſors, which exactly Divide the propos'd Equation;

you
Page 221, *l.* 15, for $=$, *r.* equal. *p.* 234, put .5 in *l.* 10, as alſo 1.5 in *l.* 11, two Places of Figures further to the Right-hand. *ibid l.* the laſt but ſeven, dele $+$. *p.* 248, *l.* 4. *r.* $\frac{20002}{30000}$. *ibid. l.* 23, dele . *p.* 225, *l.* the laſt but ten, for 1, *r.* ī.

In

ERRATA.

In the laſt Fig. in *p.* 339, for D, *r.* L. *p.* 355, *l.* 21, *r.* Example. *p.* 356, *l.* the laſt, for $\frac{1}{2}\sqrt{0}$, *r.* $\frac{1}{2}\sqrt{50}$. *p.* 386, *l.* 4, *r.* $\underline{b^4+c^4+d^4}$: . *p.* 401, *l.* 22, after *viz.* add : . *ibid l.* the laſt but one, for produc'd, *r.* produce. *p.* 407, *l.* 11, for $\because 3xx \multimap \sqsubset$, *r.* $\therefore 3xx \multimap \sqsupset$. *ib. l.* the laſt but three, for $+mx^{m-1}a$, *r.* $\pm mx^{m-1}a$. *p.* 419, *l.* 7, for infinite, *r.* finite. *p.* 423, laſt *l.* but three, *r.* $\dfrac{ca}{\sqrt{:cc+\overline{z-1}|^2 a^2:}}$. In *p.* 426, *l.* the laſt, after $3^{4\text{r}})\sqrt{12}$, place (. *p.* 430, the very laſt *l.* *r.* $5\times\overline{1}$. *p.* 438, *l.* 20, for *u*, inſert *v*. *p.* 448, *l.* 13, *r.* $\overline{b}|^{\frac{n+1}{n}}$.

A

TREA-

A
TREATISE
OF
Algebra.

BOOK I.

Definition.

THE Name *Algebra*, Dr. *Wallis* acquaints us, is derived from the firſt Word of *Algiabr Walmokabala*, which Words, in the *Arabick* Tongue, ſignifie the *Art* of *Reſtitution* and *Compariſon*, or the *Art* of *Reſolution* and *Equation*. It is a Science by which the moſt difficult and abſtruſe Queſtions, in *Arithmetick* and *Geometry*, are Reſolved and Demonſtrated; that is, it equally interferes with them both, and therefore it is promiſcuouſly Nam'd, being ſometimes call'd *Specious Arithmetick*, as by *Vieta*, *Harriot*, Dr. *Wallis*, &c. and ſometimes *Modern Geometry*, as by Dr. *Halley*.

This great *Art* may be Defined, or rather Deſcribed to be an *Analytical* Way of *Demonſtration*, where, aſſuming the Quantity or Quantities unknown, as if it, or they were known, we proceed by Conſequences, in comparing it or them, and known, or

given Quantities among themselves, until the unknown Quantity sought, or, at least, some Power or Powers thereof, be found equal to some known Quantities, and so it self is, or may be, of Consequence, made known.

CHAP. I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps, *&c.*

Sect. 1. Notation.

THE Quantity sought is called the Root, which being unknown, cannot be really express'd; but may be design'd by any Symbol, or Character at Pleasure. I commonly (with many others) use Vowels for unknown, and Consonants for known or given Quantities. But *Des Cartes* and his Followers, and most Foreign Writers, use the last Letters of the Alphabet, x, y, z, for unknown Quantities; and the first Letters, a, b, c, *&c.* for known ones.

The Advantage of thus expressing known Quantities is, that the Numbers with their several Operations, which in Vulgar Arithmetick would be lost or swallowed up, in *Algebraic* Arithmetick are so ordered and managed, as to be preserved Distinct and in View; and at last to produce a Canon or a Rule for Resolving, not only the particular Question propos'd, but in General, any other of the like Nature.

Besides these Letters, there are certain Signs used, which are as followeth.

An Explanation of the Signs used in Algebra.

$+$	Signifies	More, or added to.
$-$	Signifies	Less, or subtracted from.
$\times$	Signifies	Multiplyed by.
$\div$	Signifies	Divided by.
$=$	Signifies	Equal to.
∺	Signifies	Continual Geometrical Proportion.
::	Signifies	Disjunct Geometrical Proportion.
$\div$	Signifies	Continual Arithmetical Progression.

Greater

$\sqsubset$	Signifies	Greater than,
$\sqsupset$		Less than.
$\backsim$		The Difference of two Quantities, when 'tis not known, which of them is the Greater.
$\circledcirc$		Involution.
ɯ		Evolution, or Extracting of Roots.
$\surd$		Irrationality, or the Sign of a Surd-Root.
$\therefore$		Therefore.

$b + c$	Signifies	*b* More *c*, or *c* added to *b*.
$b - c$		*b* Less *c*, or *c* subtracted from *b*.
$b \times c$ or $b\,c$		*b* Multiplyed by *c*.
$c)b$, $b \div c$ or $\frac{b}{c}$		*b* Divided by *c*.
$b = c$		*b* Equal to *c*.
$a . b . c . d ∺$		As *a* is to *b*, so is *b* to *c*, so is *c* to *d*.
$a \cdot\cdot b :: c \cdot\cdot d$		As *a* is to *b*, so is *c* to *d*.
$a . b . c . d \div$		*a*, *b*, *c*, *d*, have equal Differences.
$b \sqsubset c$		*b* Is Greater than *c*.
$b \sqsupset c$		*b* Is Less than *c*.
$b \backsim c$		The Difference between *b* and *c*, when it is not known which of them is the Greater.
$b \circledcirc$		*b* Is to be Involved, or raised to some Power
b ɯ		*b* Is to be Evolved, or some Root to be Extracted out of it.

$\surd b$, (or $\sqrt[2]{b}$), $\sqrt[3]{b}$, $\sqrt[4]{b}$, &c. Signifie the Square-Root of *b*, the Cube-Root of *b*, the Biquadrat-Root of *b*, &c. respectively.

Besides the foregoing Signs, which are commonly used, I make use of others which are not Common, and are as follow.

⊕	Signifies	Some Quantity indefinitely Less than the Term that next precedes it, is to be added.
⊖		Some Quantity indefinitely Less than the Term which next precedes it, is to be subtracted.
⊕̸		Either ⊕ or ⊖, when it matters not which of them it is.

There are other Signs which are to be used in the Geometrical Part of Algebra, *and which will be explain'd in* Book II.

All the Quantities concern'd in any Question or Problem may stand in any Order at Pleasure, *Viz.* The most convenient for Operation: As $a + b - d$, may stand thus, $b - d + a$, or thus, $a - d + b$, or thus, $- d + a + b$, &c. these still being the same, tho' differently plac'd.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always understood to have the Sign $+$ before it, as a is $+ a$, or $b - d$ is $+ b - d$, &c; for the Sign $+$ is an Affirmative Sign, and therefore all leading, or positive Quantities are understood to have it, as well as those that are to be added.

But the Sign $-$ being a Negative Sign, or Sign of Defect, there is a Necessity of prefixing it to that Quantity to which it belongs, wherever the Quantity stands.

When any Quantity is taken more than once, you must prefix its Number to it, as $3a$ stands for three times a, and $7b$ stands for seven times b, &c.

All Numbers thus prefixt to any Quantities, are call'd Co-efficients, or Fellow-factors, because they Multiply the Quantity; and if any Quantity be without a Co-efficient, it is always suppo-sed or understood to have an Unit prefixt to it; as a is $1\,a$, or $b\,c$ is $1\,b\,c$, &c.

All Quantities that are exprefs'd in Numbers only (as in Vulgar Arithmetick) are called absolute Numbers.

Like Quantities are those which are exprefs'd by the same Letters under the same Power; as b and b, $a\,a$ and $a\,a$, $c\,d\,b$ and $6\,c\,d\,b$, &c.

Unlike Quantities are such as are exprefs'd by different Letters, or by the same Letters under different Powers; as a and b, $c\,d\,f$, and $c\,d$, b^3 and b^5, &c.

Those

Those Quantities that are represented by single Letters, as a, b, c, d, &c. or by several Letters that are immediately join'd together, as ab, cd, or $7bd$, &c. are called Simple, or single Quantities.

But when different Letters, or unlike Quantities are connected together by the Signs $+$ or $-$, as $a+b$, or $a-b$, or $ab+dc$, or $a+aa$, &c. they are called Compound Quantities.

When a Compound of divers Quantities is jointly considered as one, it is express'd, either by putting them between two Colons, or by drawing a Line over them; As $\sqrt{}:aa+ba:$ or $\sqrt{\overline{aa+ba}}$, signifies the Square-Root of $aa+ba$, consider'd as one Quantity: And $:c+d+e:\times f$, or $\overline{c+d+e}\times f$, signifies that the whole Compound Quantity $c+d+e$, is Multiplyed by f.

Algebraick Integers are such Quantities as are not express'd Fraction-wise: As a, $a+b$, &c.

But when Quantities are express'd, or set down like Vulgar Fractions, thus $\frac{a}{b}$, or $a \div b$, or $b)\,a$, or $\frac{a+d}{b}$, or $\frac{ba+de}{c-p}$, &c. They are called Fractional or broken Quantities.

Sect. 2 Of the Tracing the Steps used in bringing Quantities to an Equation.

The Method of tracing the Steps used in bringing the Quantities concern'd in any Question to an Equation, is best perform'd by Registring the several Operations with Figures and Signs plac'd in the Margin of the Work, according as the several Operations require, being very useful in long and tedious Operations.

For Instance, If it be requir'd to set down and Regisler the Sum of the two Quantities a and b, the Work will stand

Thus,	1	a
	2	b
$1+2$	3	$a+b$

First set down the propos'd Quantities a and b, over-against the Figures 1 and 2, in the small Column (which are called Steps) and against 3 (the third Step) set down the Sum, *viz.* $a+b$; then against the third Step, set down $1+2$, in the Margin, which denotes that the Quantities against the first and second Steps are added together, and that those in the third Step are their Sum.

To illustrate this in Numbers, Suppose $a=9$, and $b=6$, then it will be

Thus

$$\begin{array}{l|l|l} \text{Thus,} & 1 & a=6 \\ & 2 & b=6 \\ \hline 1+2 & 3 & a+b=9+6=15 \end{array}$$

Again, if it were required to ſet down the Difference of the ſame two Quantities, it will be

$$\begin{array}{l|l|l} \text{Thus,} & 1 & a=9 \\ & 2 & b=6 \\ \hline 1-2 & 3 & a-b=9-6=3 \end{array}$$

Or if it were required to ſet down their Product, then it will be

$$\begin{array}{l|l|l} \text{Thus,} & 1 & a=9 \\ & 2 & b=6 \\ \hline 1\times 2 & 3 & a\times b \text{ or } ab=9\times 6=54 \end{array}$$

&c.

Note, Letters ſet, or join'd immediately together, (like a Word) ſignifie the Rectangle, or Product of thoſe Quantities they repreſent; as in the laſt Example, wherein $ab=54$ is the Product of $a=9$, and $b=6$.

Axioms.

1. If equal Quantities be added to equal Quantities, the Sums of thoſe Quantities will be equal.
2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal.
3. If equal Quantities be Multiplyed by equal Quantities, the Products will be equal.
4. If equal Quantities be Divided by equal Quantities, the Quotients will be equal.
5. Thoſe Quantities that are equal to one and the ſame thing, are equal to one another.
6. Every Whole is equal to all its Parts taken together.

PART

PART I.

Of whole Quantities.

CHAP. II.

Addition of whole Quantities.

ADdition in *Algebra* may be easily learn'd, by observing the following particular *Rules* or *Cases*.

Rule 1. When Simple and like Quantities having like Signs are to be added.

Add the Co-efficients, or prefixt Numbers together, and to their Sum adjoin the Letters common to each, or in either of the said Quantities. Lastly, to this Sum prefix the common Sign, and you'll have the Sum required.

N.B. Ex. signifies *Example.*

		Ex. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
	1	$+b$	$-a$	$20bca$	$-52bc$
	2	$+b$	$-a$	$34bca$	$-29bc$
$1+2$	3	$+2b$	$-2a$	$54bca$	$-81bc$

		Ex 5.	*Ex.* 6.
	1	$654abod$	$-92zy$
	2	$382abcd$	$-24zy$
	3	$21abcd$	$-10zy$
	4	$349abcd$	$-58zy$
$1+2+3+4$	5	$1406abcd$	$-184zy$

The Reason of these Additions is evident from the Work of common Arithmetick; for suppose b *to represent* 1 *Crown, to which if I add* 1 *Crown, the Sum will be* 2 *Crowns, or* 2 b (*or* $+2$b) *as in Ex.* 1. *Or if we suppose* $-$ a *to represent the Want, or Debt of* 1 *Crown, to which if an other Want or Debt of* 1 *Crown be added, the Sum must needs be the Want or Debt of* 2 *Crowns, or* -2 a, *as in* Ex. 2. *And so for all the rest.*

Rule 2. When Simple and like Quantities having unlike Signs are to be added.

Add all the Affirmative Ones into one Sum, and all the Negative Ones into another (by *Rule* 1.); then prefix the Difference of the

the Co-efficients of these two Sums, with the Sign of the greater Sum, to the Letters common to each of the said Quantities; and you'll have the Sum required.

		Ex. 7.	*Ex.* 8.	*Ex.* 9.	*Ex.* 10.
	1	$+5a$	$-5a$	$+bc$	$-9abd$
	2	$-3a$	$+3a$	$-bc$	$+7abd$
$1+2$	3	$+2a$	$-2a$	$+0bc$, or 0	$-2abd$

Ex. 11.

	1	$+9bcdx$	$\left.\right\} = +16bcdx$
	2	$+7bcdx$	
	3	$-12bcdx$	$\left.\right\} = -18bcdx$
	4	$-6bcdx$	
$1+2+3+4$	5		$-2bcdx$

The Reason of this *Rule* is this,

All Quantities having Negative Signs, are in Nature directly contrary to such as have Affirmative Signs; and therefore will always destroy one another: Thus, If a Man have 1500 l. *in Cash, and run in Debt* 500 l. *that is, if to his Cash he add* — 500 l. *(which is the proper way to express a Debt) there will remain but* 1000 l. *for the Debt or* — 500 l. *will destroy* 500 *l. of the Cash. So also if a Man owe* 100 l. *and hath nothing to pay it, then he hath* — 100 l. *or is* 100 l. *worse than nothing; and if any one give him* 100 l. *or add* + 100 l. *to his* — 100 l. *the Sum will be nothing; but notwithstanding the Man (tho' worth nothing) will be* 100 l. *better than he was before.*

Rule 3. When unlike Quantities are to be added,

Set them down without altering their Signs; and hence will arise Compound Quantities: For unlike Quantities cannot be otherwise added but by their Signs:

		Ex. 12.	*Ex.* 13.	*Ex.* 14.
Thus,	1	$3a$	$-745cdd$	$20b$
	2	$2b$	$-934cd$	$-30c$
$1+2$	3	$3a+2b$	$-745cdd-934cd$	$20b-30c$

Ex. 15.

	1	$-20bc$
	2	$+40b$
	3	$-30c$
	4	$-59bd$
$1+2+3+4$	5	$-20bc+40b-30c-59bd$

Rule

Rule 4. When Compound Quantities are to be added,

Find the Sums of the like Quantities, by the first and second *Rules*, and then add these Sums and the unlike Quantities together, by the third *Rule*, and you'll have the Sum required.

Ex. 16.

	1	$3aa + 4bcd$
	2	$-2aa - 9bcd + 81e$
	3	$+9aa - 15bcd - 64e + gg$
	4	$+7aa + 20bcd \quad - g$
$1 + 2 + 3 + 4$	5	$+17aa \quad + 17e + gg - g$

CHAP. III.

Subtraction of whole Quantities.

SUbtraction of whole Quantities is perform'd by one general *Rule*.

Rule.

Change all the Signs of the Subtrahend, *viz.* of those Quantities which are to be Subtracted, or suppose them to be chang'd; then add all the Quantities together, as before in Addition; and their Sum will be the Remainder required.

That to add — is the same thing as to subtract + has been prov'd in Addition: But this general Rule of Subtraction supposes that to subtract — is all one as to add +, which Supposition may be thus explain'd and prov'd.

If a Man owe 10 l. *more than he is worth, then his Substance may, by what has been said in Addition, be represented by* — 10 l. *But if any one will pay that* 10 l. *for him, or, which is all one, take away the Debt of* 10 l. *or subtract the* — 10 l. *he doth him as much Service, as if he added* 10 l. *to his Substance; for, in either Case, he will be worth just nothing.*

		Ex. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$-6b$	$+8bc$
$1 - 2$	3	a	$-a$	$+14b$	$-23bc$

		Ex. 5.	*Ex.* 6.	*Ex.* 7.
	1	$9abc - 7acd$	$a + b - 54$	$bc + 2$
	2	$-6abc - 7acd$	$-3b - 75 - bc + d$	$-7bc + 9$
$1 - 2$	3	$15abc$	$a + 4b + 21 + bc - d$	$8bc - 7$

		Ex. 8.	*Ex.* 9:	*Ex.* 10.
	1	$a + b$	$7bc - 2bd + 4$	0
	2	$a - b$	$- bc - 9bd$	$2a - 4b$
$1 - 2$	3	$+ 2b$	$8bc + 7bd + 4$	$-2a + 4b$

		Ex. 11.	*Ex.* 12.	*Ex.* 13.
	1	$a + 20 - 97c$	76	b
	2	$45 + d$	$a - b - 5d + 7c$	$-c$
$1 - 2$	3	$a - 25 - 97c - d$	$76 - a + b + 5d - 7c$	$b + c$

That to subtract — is all one, as to add + may be thus prov'd:

If $-c$ be subtracted from $+b$, I say the Remainder r is $b + c$.

Demonstration.

Take any Quantity d greater than c, And

Suppose	1	$d - c = a$
$b - 1$	2	$b - d - -c = b - a$, by the Nature of Subtraction.
$2 + d$	3	$b - -c = b + d - a$, by what hath been said in (Addition.
that is	4	$r = b + d - a$
$4 + a$	5	$r + a = b + d$
* $1, 5$	6	$r + d - c = b + d$
$6 - d$	7	$r - c = b$
$7 + c$	8	$r = b + c$. Q. E. D.

The Truth of all Operations in Subtraction, where any Doubt arises, may be proved by adding the Subtrahend to the Remainder; as in common Arithmetick.

Examples.

	1	$5a$	0	$-9bc$	
	2	$-2a$	$3b$	$-6da$	*Subtrahend.*
$1 - 2$	3	$+7a$	$-3b$	$-9bc + 6da$	*Remainder.*
$2 + 3$	4	$+5a$	0	$-9bc$	*Proof.*

* Note, 1, 5 *signifies from the first and fifth Steps.*

CHAP.

CHAP. IV.

Multiplication of whole Quantities.

M*Ultiplication of whole Quantities admits of three* Cases.

Case 1. When two simple Quantities, whether like or unlike, but having like Signs, are to be Multiplied together.

First, Multiply the Co-efficients one into another, and then to the Product annex the Letters of both Quantities; so shall this new Quantity (the Sign + being understood as prefixt before it) be the true Product.

Ex. 1.

	1	a
	2	b
1×2	3	$1ab$, or ab, or ba;

for ab is $= ba$, by the 16. 7. of the * Father of the *Mathematicks's Elements.* **Euclid.*

		Ex. 2.	*Ex.* 3.	*Ex.* 4	*Ex.* 5.
	1	$-a$	$-abcd$	$5d$	$-6d$
	2	$-b$	$-fgh$	$3b$	$-7bc$
1×2	3	$+ab$	$abcdfgh$	$15db$	$42dbc$

Case 2. When the Quantities to be Multiplied are Simple and and have unlike Signs.

Join them and the Product of their Co-efficients together as before, but prefix the Sign — before them.

		Ex. 6.	*Ex.* 7.	*Ex.* 8.
	1	a	$-6d$	$67916plcd$
	2	$-b$	$+7b$	$-100dq$
1×2	3	$-ab$	$-42db$	$-6791600plcddq$

That in Algebraick Multiplication like Signs must give a Positive or an Affirmative Product, and unlike Signs a Negative one, may be thus prov'd:

Multiplication being a Compendious Method of adding together (or into one Sum) the Multiplicand so often repeated as there are Units in the Multiplicator, or Multiplyer; therefore,

When the Multiplicand is Affirmative, according as the Multiplicator is great or small, the Product will be proportionably great or small; and when the Multiplicator is 0 *(or nothing) the Product will be* 0 *too; consequently when the Multiplicator is*

greater or less than o, *to wit, Affirmative, or Negative, the Product must be proportionably greater or less than* o, viz. *Affirmative, or Negative ; wherefore,*

+ *Multiplyed by* + *must produce* + ; viz. $+b \times +c = +bc$, *or* $+cb$. *Also* + *Multiplyed by* − *must produce* − ; viz. $+b \times -c = -bc$, *or* $-cb$.

When the Multiplicand is Negative, according as the Multiplicator is great, or small, the Product will be proportionably small, or great (thus -2×2, *or the Sum of* -2 *and* -2, *or* -4, *is twice as small as* -2×1 *or* -2) ; *And when the Multiplicator is* o, *the Product will be* o *too ; consequently when the Multiplitator is greater, or less then* o ; *to wit, Affirmative or Negative, the Product must be proportionally less, or greater than Nothing* ; viz. *Negative or Affirmative : Wherefore,*

− *Multiplied by* + *must produce* − ; viz. $-b \times +c = -bc$, *or* $-cb$. *And* − *Multiplyed by* − *must produce* + ; viz. $-b \times -c = +bc$, *or* cb.

Case 3. If the Multiplyer and Multiplicand, or either of them be compound Quantities.

Then every Term of the Multiplyer must be Multiplyed into all the Terms of the Multiplicand ; and the Sum of those particular Products, will be the Product required, by *Ax.* 6. As in common Arithmetick.

Examples.

(9)	1	$a + b - d$
	2	$a - b$
$1 \times a$	3	$aa + ab - ad$
$1 \times -b$	4	$-ab - bb + bd$
$3 + 4$	5	$aa - ad - bb + bd$

(10)	1	$3aa - 7ba$
	2	$a + 8b$
$1 \times a$	3	$3aaa - 7baa$
$1 \times 8b$	4	$24baa - 56bba$
$3 + 4$	5	$3aaa + 17baa - 56bba$

I will add the following Numerical Examples, in order to make it the more plain to Beginners, that like Signs produce +, *and unlike Signs produce* −.

(11)

		(11)	(12)	(13)
	1	$9-5=4$	$9=9$	$-9=-9$
	2	$6\quad=6$	$6-3=3$	$+6-3=3$
1×2	3	$54-30=24$	$54-27=27$	$-54+27=-27$

That a Positive Quantity Multiplyed by a Negative, or a Negative Quantity Multiplyed by a Positive, produces a Negative one, may be thus prov'd.

If $+b$ be Multiplyed by $-c$, I say the Product p is $=-bc$.

Demonstration.

Put	1	$b+c=a$
$1-c$	2	$b=a-c$
$2\times b$	3	$bb=ba+b\times -c$, by the Nature of Multiplication.
that is	4	$bb=ba+p$
$1\times b$	5	$bb+bc=ba$
$5-bc$	6	$bb=ba-bc$
$4=6$	7	$ba+p=ba-bc$; that is $p=-bc$. *Q. E. D.*

That a Negative Multiplied by a Negative Quantity, produces a Positive one, may be prov'd thus.

If $-b$ be Multiplyed by $-c$, I say the Product p is $=+bc$.

Demonstration.

Put	1	$a=b+c$
$1-b$	2	$a-b=c$
$2\times -c$	3	$-ca+-b\times -c=-cc$, by the above Demonstration.
that is	4	$-ca+p=-cc$
$1\times -c$	5	$-ca=-cb-cc$
$5+cb$	6	$-ca+cb=-cc$
$4=6$	7	$-ca+p=-ca+cb$ ∴ $p=cb$. *Q. E. D.*

Note, that sometimes Products are express'd only by the Quantities to be Multiplyed with the Sign $\times$ between them, and each Compound Factor being connexed by Colons, or a Line over it: Thus the Product of a and b, is $a\times b$, and the Product of $a+x$, and $e-z$, is $:a+x:\times:e-z:$ or $\overline{a+x}\times\overline{e-z}$.

CHAP.

CHAP. V.

Division of whole Quantities.

Division in Species is the Converse, or direct contrary to that of Multiplication, and consequently perform'd by converse Operations (as in common Arithmetick.)

The general Rule of Division is this; *viz.*

Place the Divisor under the Dividend, with a Line between them, and this Fraction is the Quotient (as in common Arithmetick.)

Or place the Sign $\div$ between the Dividend and Divisor, and let the latter be to the Right-hand of it;

Or place the Sign) between the Dividend and Divisor, and let the latter be to the Left-hand of it;

Thus a being Divided by b, will give $\frac{a}{b}$, or $a \div b$, or $b)a$ for a Quotient: And $2bcg + dg$ Divided by $6gf - 3ga$, is $= \frac{2bcg + dg}{6gf - 3ga}$, or $= : 2bcg + dg : \div : 6gf - 3ga :$, *&c.*

But if any Quantity be found to be a common Multiplyer in both the Dividend and Divisor; Place the Quantity which, being Multiplyed by that common Multiplyer, will produce the Dividend over that Quantity which, being Multiplyed by the common Multiplyer will produce the Divisor, with a Line between them.

Examples.

		(3)	(4)	(5)	(6)
	1	cfe	abd	$75cd$	$ab + cdb$
	2	cb	ab	$-5c$	$a + cd$
$1 \div 2$	3	$\frac{fe}{b}$	$\frac{d}{1} = d$	$-15d$	b

		(7)
	1	$3ab + 15bc - 21bd$
	2	$-3b$
$1 \div 2$	3	$-a - 5c + 7d$

Note,

Note, When the Quantities in the Divisor and Dividend are all the same, and have the same Signs, the Quotient will be an Unit or 1; *because every thing contains it self once*:

So $b \div b = 1$; also: $a + b : \div : a + b : = 1$, and $\frac{a-b}{a-b} = 1$.

N.B. *In Division of Quantities one thing must be carefully observed,* viz. *That like Signs give* $+$, *and unlike Signs give* $-$, *which holds here as well as in Multiplication, as may be prov'd thus.*

Every Dividend being a Product made by Multiplying the Quotient into the Divisor, the Sign of each Factor must be such, as according to the former Rules of Multiplycation, can produce the Dividend: Wherefore if the Dividend be positive, the Divisor and Quotient must have like Signs; that is, if the Divisor be Negative, the Quotient must be so too; and of consequence a positive Dividend divided by a negative Divisor, gives a negative Quotient. And if the Dividend be Negative, the Divisor and Quotient must have unlike Signs (as has been Demonstrated in Multiplication); and consequently a negative Dividend divided by a positive Divisor, gives a negative Quotient; but a negative Dividend divided by a negative Divisor, gives an affirmative or positive Quotient.

In Division of Compound Quantities, the Terms of, as well the Dividend as Divisor, must be plac'd in order, according to the Dimensions of some Letter in both of them, which will be judg'd the most Commodious for the Purpose; that is to say, the first, second, third, &c. Terms of the Dividend, and of the Divisor, must be those which contain the greatest, greatest but one, greatest but two, &c. power of the said Letter respectively.

Then seek such a Quantity, as being Multiplyed by the first Term of the Divisor, shall produce the first Term of the Dividend, which Quantity, when found, place in the Quotient, and then Multiply it into the whole Divisor, the Product subtract from the respective Terms of the Dividend; and to the Remainder adjoin, with their proper Signs, as many more of the next following Terms of the Dividend as are requisite, and call this Sum your Dividual. Again, seek another Term of the Quotient, *viz.* Such as being Multiplyed by the first Term of the Divisor, will produce the first Term of your Dividual, which Term, when found, place in the Quotient; then Multiply, and Subtract as aforesaid; and so proceed.

Thus, If $aaa + 3aae + 3aee + eee$ be given to be Divided by $a + e$, I place them in this Manner,

$$\frac{aaa + 3aae + 3aee + eee}{a + e}$$

Then

Then I seek such a Quantity as, being Multiplyed by a (the first Term of the Divisor) will produce aaa (the first Term of the Dividend) and finding it to be aa, I place aa in the Quotient, then I Multiply $a + e$ by aa, and the Product $aaa + aae$ I subtract from $aaa + 3aae$ (the respective Terms of the Dividend) and the Remainder is $2aae$; to which I adjoin the next Term of the Dividend, *viz.* $+ 3aee$, and the Sum $2aae + 3aee$ is my Dividual.

Again I seek a Quantity, which, being Multiplyed by a (the first Term of the Divisor) will produce $2aae$ (the first Term of my Dividual) and find it to be $+ 2ae$, which $+ 2ae$ I place in the Quotient, then I Multiply it by (or into) $a + e$, and the Product $2aae + 2aee$, I subtract from $2aae + 3aee$, and the Remainder is aee; to which I adjoin the next and last Term of the Dividend, *viz.* $+ eee$, and the Sum $aee + eee$ is my Dividual.

Then I seek a Quantity which, being Multiplyed by a, will produce aee (or ask how oft a is contained in aee) and finding it to be ee, I place $+ ee$ in the Quotient, and Multiply $a + e$ by it, and the Product $aee + eee$ I Subtract from my last Dividual, and the Remainder is nothing. See the following Operation.

Example II.

		Quotient
	1	$aaa + 3aae + 3aee + eee\ (aa + 2ae + ee$
	2	$a + e$
$2 \times aa$	3	$aaa + aae$
$aaa + 3aae - 3$	4	$0 + 2aae$
$4 + 3aee$	5	$2aae + 3aee$
$2 \times 2ae$	6	$2aae + 2aee$
$5 - 6$	7	$0 + aee$
$7 + eee$	8	$aee + eee$
$2 \times ee$	9	$aee + eee$
$8 - 9$	10	0

Or Division of compound Quantities may be better perform'd thus.

Ex. 12.

$$
\begin{array}{l}
3a - 6)\ 6aaaa - 96 \quad (2aaa + 4aa + 8a + 16 \\
\quad\quad\ \ \underline{6aaaa - 12aaa} \\
\quad\quad\quad\ \ 0 + 12aaa \\
\quad\quad\quad\quad\quad \underline{12aaa - 24aa} \\
\quad\quad\quad\quad\quad\quad\quad + 24aa \\
\quad\quad\quad\quad\quad\quad\quad\ \ \underline{24aa - 48a} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad + 48a - 96 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ \underline{48a - 96} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

Suppose it were required to divide $aaa + 4caa + daa + 4cda + dda + ddd$ by $a + d$.

If aaa be made the first Term of the Dividend, $+ 4caa + daa$, $+ 4cda + dda$, $+ ddd$ must be the second, third, and fourth Terms of it respectively; and a, $+ d$ the first and second Terms of the Divisor respectively: And then the Division will stand thus,

$$
\begin{array}{l}
a + d)\ aaa \begin{array}{l} + 4caa \\ + daa \end{array} \begin{array}{l} + 4cda \\ + dda \end{array} + ddd \quad (aa + 4ca + dd \\
\quad\quad \underline{aaa + daa} \\
\quad\quad\quad 0 + 4caa \begin{array}{l} + 4cda \\ + dda \end{array} \\
\quad\quad\quad\quad\ \ \underline{4caa + 4cda} \\
\quad\quad\quad\quad\quad\quad + dda + ddd \\
\quad\quad\quad\quad\quad\quad\quad \underline{dda + ddd} \\
\quad\quad\quad\quad\quad\quad\quad\quad\ \ 0
\end{array}
$$

N. B. The foregoing Dividend may be writ thus,

$$aaa \begin{array}{l} + 4c \\ + d \end{array} aa \begin{array}{l} + 4cd \\ + dd \end{array} a + ddd$$

Here the Members of the second Term, as also those of the third are united by adding in each Term the Factors of the Powers of the Letter a, in respect of which the Terms of the Dividend were plac'd.

D But

But if ddd be made the first Term of the Dividend, then the Division will stand thus,

$$d+a)\ ddd + add \begin{matrix}+aa\\+4ca\end{matrix} d \begin{matrix}+aaa\\+4caa\end{matrix} \left(dd \begin{matrix}+aa\\+4ca\end{matrix}\right.$$

$$\begin{array}{r}
ddd + add \qquad\qquad\qquad\qquad \\
\hline
0 \begin{matrix}+aa\\+4ca\end{matrix} d \begin{matrix}+aaa\\+4caa\end{matrix} \\
\begin{matrix}aa\\+4ca\end{matrix} d \begin{matrix}+aaa\\+4caa\end{matrix} \\
\hline
0 \qquad
\end{array}$$

If the Divisor be not an Aliquot Part of the Dividend, the Quotient may, in some Cases, be continued to an Infinite Series: But if, after you have plac'd as many Terms in the Quotient as you think proper, you have a Mind to have the exact Quotient, place the last Remainder over your Divisor, with a Line between them, which Fraction with its proper Sign + or — annexed to the before found Quotient, will exhibit the exact Quotient required.

Examples.

$$1-x)\ 1 \quad \left(\begin{array}{l} 1 + x + xx + xxx + \text{\&c. } \textit{Sine Fine.} \\ \text{Or } 1 + x + xx + xxx + \dfrac{xxxx}{1-x} \end{array}\right.$$

$$\begin{array}{l}
1 - x \\
\hline
0 + x \\
\quad x - xx \\
\hline
\qquad + xx \\
\qquad\quad xx - xxx \\
\hline
\qquad\qquad + xxx \\
\qquad\qquad\quad xxx - xxxx \\
\hline
\qquad\qquad\qquad + xxxx
\end{array}$$

$aa - ee$

$$aa - ee \,)\, aae \,\left(\, e + \frac{eee}{aa} + \frac{eeeee}{aaaa} + \&c,\right.$$

$$aae - eee$$

$$+\,eee$$

$$eee - \frac{eeeee}{aa}$$

$$+\,\frac{eeeee}{aa}$$

$$\frac{eeeee}{aa} - \frac{eeeeeee}{aaaa}$$

$$+\,\frac{eeeeeee}{aaaa}\ \text{Remainder.}$$

PART II.

Of Fractional or Broken Quantities.

CHAP. I.

Notation of Fractionrl Quantities.

FRactional Quantities are expreſs'd, or ſet down, like Vulgar *Fractions* in common Arithmetick,

$$\text{Thus}\left\{ \frac{a}{b},\ \frac{5b-4a}{4d+7b},\ \frac{2bc}{d} \right. \quad \begin{matrix}\textit{Numerators.}\\ \textit{Denominators.}\end{matrix}$$

How they came to be ſo, may be ſeen by the General Rule in the Beginning of Diviſion.

Theſe *Fractional Quantities* are manag'd in *Algebra*, as broken Numbers in *Arithmetick*.

CHAP. II.

Reduction of Fractional Quantities.

Sect. 1. To reduce Fractions having different Denominators, to Fractions of the ſame Value, that ſhall have a common Denominator.

Rule.

MUltiply all the Denominators together, and reſerve their Product for a new and common Denominator; then Multiply any of the Numerators, and all the Denominators, but its own, together, and their Product put for a Numerator over the common Denominator; ſo this Fraction is equal to that whoſe Numerator you Multiplyed. Do ſo with the reſt of the Numerators, and you'll have the Fractions required.

Examples

Examples.

1. Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ to one Denomination.

First, $b \times c = bc$, is the common Denominator.

Secondly, $a \times c = ac$; therefore $\frac{ac}{bc}$ is $= \frac{a}{b}$.

Thirdly, $b \times d = bd$; therefore $\frac{bd}{bc} = \frac{d}{c}$.

Consequently $\frac{ac}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required.

2. Let $\frac{cdf}{e}$, $\frac{ab}{fd}$ and $\frac{a+d}{c-d}$ be, all of them, brought to a common Denominator.

First, $e \times fd \times :c-d: = efdc - efdd$ is the common Denominator.

Secondly, $cdf \times fd \times :c-d: = ccddff - cdddff$; $ab \times e \times :c-d: = abec - abed$; and $:a+d: \times e \times fd = aefd + ddef$, are the Numerators;

Therefore $\frac{ccddff - cdddff}{efdc - efdd}$, $\frac{abec - abed}{efdc - efdd}$, and $\frac{aefd + ddef}{efdc - efdd}$, are the Fractions required.

3. If $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought to one Denomination, they will be $\frac{bb+bc-bd-dc}{ba+bb-da-db}$ and $\frac{ad-ac+bd-bc}{ba+bb-da-db}$ respectively.

Sect. 2. To reduce a whole Quantity into an Equivalent Fraction of a given Denomination.

Rule.

Multiply the whole Quantity by the given Denominator, and under the Product place the said Denominator with a Line between them; and you will have the Fraction required.

Exam-

Examples.

1. Let it be required to bring $a+b$ into a Fraction, whose Denominator shall be $d-a$.

First: $a+b : \times : d-a := da+db-aa-ba$,

Then $\frac{da+db-aa-ba}{d-a}$ is the Fraction required.

2. b being reduc'd to a Fraction, whose Denominator will be $q-r$ is $=\frac{bq-br}{q-r}$.

Note, *When whole Quantities are to be set down Fraction-wise;* subscribe an Unit for the Denominator; thus ab is $=\frac{ab}{1}$, and $b+aa=\frac{b+aa}{1}$.

Lemma to *Sect.* 3.

How to find the greatest common Divisor (or greatest common Measure) of two propos'd whole Quantities.

Rule.

Divide the greater propos'd Quantity by the lesser, and, if any thing remains, Divide your Divisor thereby, and, if any thing yet remains, Divide your last Divisor thereby; and thus proceed 'till nothing remains, provided the first Term of each Remainder measures the first Term of the next foregoing Divisor; and the last Divisor is the greatest common one required. But here, *Note*, that when the first Term of any Divisor (or Remainder) does not measure the first Term of its respective Dividend (or next foregoing Divisor), Then find a Measure of such Divisor, which Measure you are to consider three ways; *viz.*

1. As being not Prime to the Dividend; or $\left\{\begin{matrix}2\\3\end{matrix}\right\}$ as being Prime to the Dividend, and that, upon Dividing the Divisor by it, the first Term of the Quotient shall $\left\{\begin{matrix}\text{not be}\\\text{be}\end{matrix}\right\}$ a Measure of the first Term of the Dividend.

In *Case* 1. First find (by Inspection, or by what is said here) the greatest common Measure of the Divisor's said Measure and of the Dividend: Then Divide the Dividend by the said greatest common Measure, and the Divisor by its said Measure, and the

the two Quotients will be your next Dividend and Divisor, which are in less Terms than the former, and with which proceed further after the same Manner, until you arrive at such a Divisor as will Divide its respective Dividend without leaving a Remainder; which last Divisor Multiplyed continually into all the greatest common Measures, found as aforesaid, will produce the greatest common Divisor required.

In *Case* 2. Divide the Divisor by its said Measure, and, with the Quotient, as a Divisor, and the said Dividend, proceed as herein directed. Note, *the Measure, in this 2d Case, is of no great Value.*

In *Case* 3. Divide the Divisor by its said Measure; then, by the Quotient Divide the said Dividend; and, if there be a Remainder it will be your next Divisor, and the last Quotient your next Dividend, with both of which proceed as herein directed. And, if there be no greatest common Measure, such as is mention'd in *Case* 1. the last Divisor, thus found, is the greatest common one required.

The Demonstration of this 3d Case is inserted in the first Edition of this Treatise, which may be easily extended to the 1st and 2d Cases.

Sect. 3. To Abbreviate, or bring a Fractional Quantity into its lowest Denomination.

Rule.

Divide the Numerator and Denominator severally by their greatest common Divisor, and the respective Quotients, plac'd Fraction-wise, is the Fraction required.

Examples.

1. Let it be required to reduce $\frac{560\,bc}{189\,b}$ to its least Terms.

First, in order to find the greatest common Divisor of $560\,bc$ and $189\,b$, I proceed according as the Rule in the foregoing *Lemma* directs: Thus, *viz.* I Divide the greater of the two propos'd Quantities, to wit $560\,bc$ by the lesser $189\,b$; and (the Quotient being $2\,c$) the Remainder is $182\,bc$, which is to be my next Divisor, and $189\,b$ (the former Divisor) my next Dividend. But $182\,bc$ does not measure $189\,b$; wherefore, finding $7\,bc$ to be a Measure of $182\,bc$, as also $7\,b$ to be the greatest common Measure of $7\,bc$ and $189\,b$, I Divide $189\,b$ by $7\,b$, and $182\,bc$ by $7\,bc$, and the Quotients are 27 and

26;

26; which are my next Dividend and Divisor. But, these being Prime to one another, the last Divisor will be 1: Consequently the above found greatest common Measure, *viz.* $7b$ Multiplyed by $1 = 7b$ is the greatest common Divisor required.

Then I proceed by the above Rule; thus

$$\begin{array}{l} 7b)\ 560bc\ (80c \\ 7b)\ 189b\ \ (27 \end{array}\quad \frac{80c}{27} \text{ which is the Fraction required.}$$

Note, *The greatest common Divisor of Simple Quantities may be more easily got by finding the greatest common Measure of the Letters in both Quantities (which is done by Inspection only); as also that of the Numbers: And then these greatest common Measures, Multiplyed together, will produce that which is required.*

2. Let it be required to reduce to its lowest Terms

$$\frac{c^4 \begin{array}{c}-aa\\-dd\end{array} cc + aadd}{2c^3 - 2acc - 4adc + 4aad}.$$

The Terms of this Fraction are dispos'd according to the Dimensions of the Letter c. And finding the first Term of the Denominator (which I suppose to be the lesser Quantity) is not a Measure of the first Term of the Numerator; I may Divide the Denominator by 2 a Measure thereof which will give the first Term of the Quotient a Measure of the first Term of the Numerator; and then proceed: Or rather (in order to abridge the Operation), finding $2c - 2a$ to be a Measure of the Denominator, as also $c - a$ to be the greatest common Measure of the Denominator's said Measure and of the Numerator, I Divide the Numerator by the said greatest common Measure, and the Denominator by its said Measure, and the Quotients are $c^3 + acc - ddc - add$ and $cc - 2ad$, which are my next Dividend and Divisor respectively; with which proceeding, by our Rule, they will be found to be Prime to one another; and consequently the last Divisor will be 1: Wherefore $:c - a: \times 1 = c - a$ is the greatest common Measure by which the propos'd Fraction will be reduc'd to $\frac{c^3 + acc - ddc - add}{2cc - 4ad}$ by this *Sect.* as was required.

3. Let it be required to reduce $\frac{cc - 1}{ccc - 1}$ to its lowest Terms.

First

First I divide the greater of the two propos'd Quantities, *viz.* $ccc - 1$ by the lesser $cc - 1$; thus,

$$\begin{array}{lll} cc - 1) & ccc - 1 & (c \\ & \underline{ccc - c} & \\ & \quad + c - 1 & \end{array}$$

Now since the first Term of the Divisor has a greater Power of c in it than the first Term of $c - 1$, this therefore is a Remainder; wherefore I divide the Divisor by it; thus,

$$\begin{array}{lll} c - 1) & cc - 1 & (c + 1 \\ & \underline{cc - c} & \\ & + c - 1 & \\ & \underline{\quad c - 1} & \end{array}$$

And the Remainder is 0

Whence $c - 1$ is the greatest common Measure sought, by which the propos'd Fraction will be reduc'd to its lowest Terms; thus,

$$\begin{array}{lll} c - 1) & cc - 1 & (c + 1 \text{ the new Numerator.} \\ & \underline{cc - c} & \\ & + c - 1 & \\ & \underline{\quad c - 1} & \\ & \quad\ 0 & \end{array}$$

$$\begin{array}{lll} c - 1) & ccc - 1 & (cc + c + 1 \text{ the new Denominator.} \\ & \underline{ccc - cc} & \\ & + cc & \\ & \underline{\quad cc - c} & \\ & \quad + c - 1 & \\ & \quad \underline{\ \ c - 1} & \\ & \qquad 0 & \end{array}$$

Consequently $\frac{c + 1}{cc + c + 1}$ is the Fraction required.

4. Let it be required to reduce $\frac{aaaa - bbbb}{aaa - 3aab + 3abb - bbb}$ to its lowest Terms.

First I Divide the greater of the two propos'd Quantities, *viz.* that which has the greatest Power of a in't (in respect of which Letter the Terms were plac'd) by the Lesser; thus,

$$\begin{array}{l}
aaa-3aab+3abb-bbb)\,aaaa-bbbb\,(a+3b \\
\qquad\qquad\qquad\qquad\quad aaaa-3aaab+3aabb-abbb \\
\hline
\qquad\qquad\qquad\qquad\qquad +3aaab-3aabb+abbb-bbbb \\
\qquad\qquad\qquad\qquad\qquad\ \ 3aaab-9aabb+9abbb-3bbbb \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad +6aabb-8abbb+2bbbb
\end{array}$$

Now since the first Term of the Divisor has a greater Power of a in it than the first Term of $6aabb-8abbb+2bbbb$, this therefore is a Remainder; but the first Term thereof, *viz.* $6aabb$ does not Measure aaa the first Term of the Divisor; wherefore I divide the said Remainder by $6abb-2bbb$ a Measure thereof, which is Prime to the Divisor, and will give the first Term of the Quotient a Measure of the first Term of the Divisor; thus,

$$\begin{array}{l}
6abb-2bbb)\,6aabb-8abbb+2bbbb\,(a-b \\
\qquad\qquad\quad\ 6aabb-2abbb \\
\hline
\qquad\qquad\qquad\quad -6abbb+2bbbb \\
\qquad\qquad\qquad\quad -6abbb+2bbbb \\
\hline
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

And the Quotient is $a-b$; by which I Divide the said Divisor; thus,

$$\begin{array}{l}
a-b)\,aaa-3aab+3abb-bbb\,(aa-2ab+bb \\
\qquad\quad aaa-\ \ aab \\
\hline
\qquad\qquad\ -2aab+3abb \\
\qquad\qquad\ -2aab+2abb \\
\hline
\qquad\qquad\qquad\qquad abb-bbb \\
\qquad\qquad\qquad\qquad abb-bbb \\
\hline
\end{array}$$

And the Remainder is 0

Whence $a-b$ is the greatest common Measure sought, by which the propos'd Fraction will be reduc'd to $\dfrac{aaa+aab+abb+bbb}{aa-2ab+bb}$ by this *Sect.*

5. In

5. In like Manner $\frac{xxxx - bbbb}{xxxxx - xxxbb}$ will be reduc'd to its lowest Terms; thus,

$$xxxx - bbbb)\ xxxxx - xxxbb\ (x$$
$$\underline{xxxxx - xbbbb}$$
$$- xxxbb + xbbbb \text{ Remainder.}$$

$$-bb) - xxxbb + xbbbb\ (xxx - xbb)\ xxxx - bbbb\ (x$$
$$\underline{-xxxbb} \qquad\qquad \underline{xxxx - xxbb}$$
$$0 \quad + xbbbb \qquad\qquad + xxbb - bbbb \text{ Remainder.}$$
$$\underline{+ xbbbb}$$
$$0$$

$$bb)\ xxbb - bbbb\ (xx - bb)\ xxx - xbb\ (x$$
$$\underline{xxbb} \qquad\qquad \underline{xxx - xbb}$$
$$0 \quad - bbbb \qquad\qquad 0 \quad \text{Remainder.}$$
$$\underline{- bbbb}$$
$$0$$

Hence $xx - bb$ is the greateſt common Diviſor of $xxxx - bbbb$ and $xxxxx - xxxbb$, by which $\frac{xxxx - bbbb}{xxxxx - xxxbb}$ will be reduc'd to $\frac{xx + bb}{xxx}$ by this *Sect.*

If by proceeding in the foregoing Manner, you find the greateſt common Meaſure to be 1; then the Fraction is in its leaſt Terms already.

Note, *Some Fractional Quantities are reduc'd to their leaſt Terms by Dividing the Numerator and Denominator ſeverally by the greateſt common Meaſure of their ſeveral Members; thus,* $\frac{49ed - 42cd + 294bd}{35qbd + 28dc}$ *is reduc'd to* $\frac{7e - 6c + 42b}{5qb + 4c}$ *by Dividing the propos'd Numerator and Denominator ſeverally by* 7d *the greateſt common Diviſor of their ſeveral Members.*

Sect. 4. To reduce a Compound Fraction to a Simple one of the ſame Value.

Rule.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; ſo this new Fraction is that required.

Examples.

1. Let it be required to reduce $\frac{3}{b}$ of $\frac{1}{c}$ of $\frac{a}{d}$ to a simple Fraction of the same Value.

First $3 \times 1 \times a = 3a$ the new Numerator.
Secondly $b \times c \times d = bcd$ the new Denominator.

Consequently $\frac{3a}{bcd}$ is the new Fraction required.

2. Let it be required to reduce $\frac{a+b}{f}$ of $\frac{1}{c+d}$ of $\frac{1}{p}$ of $\frac{q}{r}$ to a single Fraction of the same Value.

$:a+b: \times 1 \times 1 \times q = aq + bq$ the Numerator.
$f \times :c+d: \times p \times r = fcpr + fdpr$ the Denominator.

Consequently $\frac{aq+bq}{fcpr+fdpr}$ is the simple Fraction required.

CHAP. III, and IV.

Addition and Subtraction of Fractional Quantities.

What hath been done by the Rules in the foregoing Chapter, is chiefly to fit and prepare Fractions of different Denominations for Addition, or Subtraction as Occasion requires; viz.

1. If the Fractions to be Added, or Subtracted, be Compound ones, they must be reduc'd to simple or pure Fractions (by *Sect.* 4. of the foregoing Chap.)

2. If they have not a common Denominator, they must be reduc'd to Fractions of the same Value that will have a common Denominator (by *Sect.* 1. Chap. 2.)

That being done, Addition and Subtraction are thus perform'd.

Rule.

Add or Subtract their Numerators, as Occasion requires; and under their Sum or Remainder subscribe the common Denominator, with a Line between them.

Examples

Examples in Addition.

	1	$\frac{a+b}{d}$	$\frac{2a+b}{d+c}$	$\frac{a-b+d}{d+a}$
	2	$\frac{aa}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+a}$
1+2	3	$\frac{a+b+aa}{d}$	$\frac{a+3b}{d+c}$	$\frac{2a}{d+a}$

4. Suppose it was required to add $b+\frac{c}{d}$, $f-\frac{q}{p}$ and $g+\frac{c}{p}$ into one Sum.

First, the Fractional Parts are $\frac{c}{d}$, $-\frac{q}{p}$ and $+\frac{c}{p}$; Secondly, $d\times p\times p=dpp$ is the common Denominator. Thirdly, $c\times p\times p=cpp$; $-q\times d\times p=-qdp$, and $c\times d\times p=cdp$ are the Numerators;

Wherefore $\frac{cpp-qdp+cdp}{dpp}=\frac{cp-qd+cd}{dp}$ (by *Sect.* 3. Chap. 2.) is the Sum of the Fractional Parts; consequently the Sum required, is $b+f+g+\frac{cp-qd+cd}{dp}$.

5. Again, Let it be required to add $\frac{1}{b}$ of $\frac{d}{p}$ to $\frac{d-q}{pb}$.

First $1\times d=d$; Secondly $b\times p=bp$; therefore $\frac{1}{b}$ of $\frac{d}{p}$ is (by *Sect.* 4. Chap. 2.) $=\frac{d}{bp}$: Consequently $\frac{2d-q}{bp}$ is the Sum required.

Examples in Subtraction.

	1	$\frac{a+b+aa}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$
	2	$\frac{a+b}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+a}$
1−2	3	$\frac{aa}{d}$	$\frac{2a-b}{d+c}$	$\frac{a-b+d}{d+a}$

4. Let

4. Let it be required to take $a + \frac{d}{p}$ from $\frac{a+c}{b+d}$.

First, $a + \frac{d}{p}$ is (by *Sect.* 2. Chap. 2.) $= \frac{ap+d}{p}$.

Secondly, $\frac{ap+d}{p}$ and $\frac{a+c}{b+d}$ are (by *Sect.* 1. Chap 2.) equal to $\frac{bap+bd+dap+dd}{pb+pd}$ and $\frac{pa+pc}{pb+pd}$ respectively.

Consequently $\frac{pa+pc-bap-bd-dap-dd}{pb+pd}$ is the Remainder required.

5. Again, Let it be required to take $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{b}{a}$ from $\frac{b}{a}$.

$\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{b}{a}$ is (by *Sect.* 4. Chap. 2.) $= \frac{3b}{8a}$;
And $\frac{b}{a}$ is $= \frac{8b}{8a}$; therefore $\frac{8b-3b}{8a} = \frac{5b}{8a}$ is the Remainder sought.

Note, *The Universal Method of Adding and Subtracting either whole or fracted Quantities is by* + *and* — *respectively.*

CHAP. V.

Multiplication of Fractional Quantities.

FIrst prepare mix'd Quantities (if there be any to be Multiplyed) by reducing them to Fractions of the same Denomination (by *Sect.* 1. Chap. 2.), and whole Quantities by Subscribing an Unit under them; then,

Rule.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; then this new Fraction is the Product required.

Examples.

Examples.

	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1×2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

3. Suppose it was required to Multiply $2a+\frac{b}{c}-25$ by $3b+4c$.

These prepar'd for the Work, as above directed, will stand thus,

	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
1×2	3	$\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$

N. B. *Any Fraction is Multiplyed by its Denominator by casting off, or taking away the Denominator.*

Thus $\frac{b}{a}\times a$ is $=b$; for $\frac{b}{a}\times\frac{a}{1}=\frac{ba}{a}=b$.

4. Again, Let it be required to Multiply $\frac{b}{c}$ of $\frac{a+b}{b+c}$ by f.

$$\left.\begin{array}{l} \text{First, } b\times:a+b:\times f=\underline{baf+bbf}, \text{ the Numerator,} \\ \text{Sec. } c\times:b+c:\times 1=cb+cc, \text{ the Denomin.} \end{array}\right\}\text{ of, or}$$

the Product required.

CHAP.

CHAP. VI.

Division of Fractional Quantities.

THE Fractional Quantities being prepar'd, as directed in the last Chapter; then,

Rule.

Multiply the Numerator of the Dividend, by the Denominator of the Divisor, for a new Numerator; and Multiply the Denominator of the Dividend by the Numerator of the Divisor, for a new Denominator; so this new Fraction is the Quotient required.

Examples.

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$ the Work may stand thus, $\frac{ab}{c}\Big)\frac{abd}{cf}\Big(\frac{abcd}{abef}=\frac{d}{f}$ (by *Sect.* 3. Chap. 2.)

Or thus		
	1	$\frac{abd}{cf}$
	2	$\frac{ab}{c}$
$1 \div 2$	3	$\frac{abcd}{abcf}=\frac{d}{f}$

	1	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{c-b}{a}$	$\frac{aa-ba+bb}{c}$
$1 \div 2$	3	$\frac{aa+ab}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppose

Suppose it were required to divide $aa + \frac{3bba}{a+4b}$ by $a+b$. The Work (when prepar'd as before directed) will stand thus,

$$\frac{a+b}{1}\Big)\ \frac{aaa+4aab+3abb}{a+4b}\ \Big(\frac{aaa+4aab+3abb}{aa+5ab+4bb}:$$

And $\frac{aaa+4aab+3abb}{aa+5ab+4bb}$ is $=\frac{aa+3ab}{a+4b}$, by Sect. 3. Chap. 2.

When Fractions are of one Denomination, reject the Denominators, and divide one Numerator by the other.

Thus, If $\frac{abbb}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb)\ abbb\ ($ ab the Quotient required.

For $\frac{bb}{c}\Big)\ \frac{abbb}{c}\ \Big(\frac{abbbc}{bbc}=ab.$

Again, suppose it was required to divide $\frac{aaa-abb}{c-d}$ by $\frac{aa+2ab+bb}{c-d}$.

Rejecting $c-d$ in both, it will be

$aa+2ab+bb)\ aaa-abb\ \Big(\frac{aaa-abb}{aa+2ab+bb}=\frac{aa-ab}{a+b}$ (by Sect. 3. Chap. 2.)

PART III.

Involution.

CHAP. I.

Involution of whole Quantities.

INvolution is the Raising or Producing of Powers, from any propos'd Root, and is perform'd by Multiplication.

Examples.

	1	a	$-a$	*the Root, or single Power.*
1 ⊙ 2	2	aa	$+aa$	*Square, or second Power.*
1 ⊙ 3	3	aaa	$-aaa$	*Cube, or third Power.*
1 ⊙ 4	4	$aaaa$	$+aaaa$	*Biquadrat, or 4th Power.*
1 ⊙ 5	5	$aaaaa$	$-aaaaa$	*Surfolid, or 5th Power.* &c.

Note, The Figures plac'd in the Margin after the Sign (⊙) of Involution, shew to what Height the Root is Involved; and are call'd Indices of the Power; and are usually plac'd over the Involved Quantities, in order to contract the Work, especially when the Powers are of high Dimensions.

$$\text{Thus}\left\{\begin{array}{l} a^1 = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{array}\right. \qquad \text{and}\left\{\begin{array}{l} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5\, b^6 = aaaaabbbbbb \\ a^2\, d^3\, b^4 = aadddbbbb \end{array}\right.$$

If the Quantities have Co-efficients, the Co-efficients must be Involved along with the Quantities. As in these,

Thus

Thus	1	$2a$	$-\ 3a$	$5bc$
$1 \odot 2$	2	$4aa$	$+\ 9aa$	$25bbcc$
$1 \odot 3$	3	$8aaa$	$-\ 27aaa$	$125bbbccc$
$1 \odot 4$	4	$16aaaa$	$+\ 81aaaa$	$625bbbbcccc$
$1 \odot 5$	5	$32aaaaa$	$-\ 243aaaaa$	$3125b^5c^5$. &c.

Involution of Compound Quantities is performed in the same Manner, due Regard being had to their Signs and Co-efficients.

As for Instance, Suppose $a+b$ were given to be Involved to the 5th Power.

Thus	1	$a+b$ called a *Binomial Root.*
		$a+b$
$1 \times a$	2	$aa+ab$
$1 \times b$	3	$+ab+bb$
$2+3$	4	$aa+2ab+bb$ the *Square* of $a+b$.
		$a+b$
$4 \times a$	5	$aaa+2aab+abb$
$4 \times b$	6	$aab+2abb+bbb$
$5+6$	7	$aaa+3aab+3abb+bbb$ the *Cube* of $a+b$.
		$a+b$
$7 \times a$	8	$aaaa+3aaab+3aabb+abbb$
$7 \times b$	9	$+aaab+3aabb+3abbb+bbbb$
$8+9$	10	$aaaa+4aaab+6aabb+4abbb+bbbb$
		$a+b$
$10 \times a$	11	$a^5+4a^4b+6a^3bb+4aab^3+ab^4$
$10 \times b$	12	$a^4b+4a^3bb+6aab^3+4ab^4+b^5$
$11+12$	13	$a^5+5a^4b+10a^3bb+10aab^3+5ab^4+b^5$ the *Sursolid*, or the 5th Power of $a+b$ required.

Again, Let $a-b$ a Residual Root be given to be Involved.

Then	1	$a-b$
		$a-b$
$1 \times a$	2	$aa-ab$
$1 \times -b$	3	$-ab+bb$
$1 \odot 2$	4	$aa-2ab+bb$
		$a-b$

$4 \times a$	5	$a^3 - 2aab + abb$
$4 \times -b$	6	$\quad - aab + 2abb - bbb$
1 @ 3	7	$a^3 - 3aab + 3abb - bbb$
		$a - b$
$7 \times a$	8	$a^4 - 3a^3b + 3aabb - ab^3$
$7 \times -b$	9	$\quad - a^3b + 3aabb - 3ab^3 + b^4$
1 @ 4	10	$a^4 - 4a^3b + 6aabb - 4ab^3 + b^4$
		$a - b$
$10 \times a$	11	$a^5 - 4a^4b + 6a^3bb - 4aab^3 + ab^4$
$10 \times -b$	12	$\quad - a^4b + 4a^3bb - 6aab^3 + 4ab^4 - b^5$
1 @ 5	13	$a^5 - 5a^4b + 10a^3bb - 10aab^3 + 5ab^4 - b^5$
		the 5th *Power* of $a - b$. &c.

By comparing these two Examples together, you may make the following Observations.

1. That the Powers rais'd from a Residual Root (*viz.* the Difference of two Quantities) are the same with their like Powers raised from a Binomial Root (or the Sum of two Quantities) save only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term; but the Residual Powers have the Signs + and − interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity (a) continually decrease in Arithmetical Progression; *viz.* in the Square it is a^2, a^1; In the Cube a^3, a^2, a^1; In the Biquadrat a^4, a^3, a^2, a^1; &c.

3. The Indices of the other Quantity b, do continually increase in Arithmetical Progression; *viz.* in the Square it is b^1, b^2; In the Cube b, b^2, b^3; In the Biquadrat b, b^2, b^3, b^4; &c.

4. The first and last Terms are always pure Powers of the single Quantities, and are both of the same Height.

5. The Sum of the Indices of any two Letters join'd together in the intermediate Terms, are always equal to the Index of the highest Power, *viz.* of the first or last Term.

These Observations being duly consider'd, it will be easy to conceive how the Terms of any propos'd Power rais'd from a Binomial or Residual Root, must stand without their *Unciæ*, or Numeral Figures, or Co-efficients.

For Instance, Suppose it were required to raise the Binomial Root $a + b$ to the 7th Power; then the Terms of that Power will stand without their *Unciæ* in this Order;

Viz. $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$.

And

And becauſe the *Unciæ* (not only of any ſingle Letter, but alſo) of every ſingle Power, how high ſoever it be, is an Unit or 1 (which neither Multiplies nor Divides) and all the Powers of any Binomial or Reſidual are naturally rais'd by Multiplying of the precedent Power into its original Root, which is done by only joining each Letter in the Root to the precedent Power with its *Unciæ*, and then removing the ſaid Power, when it is ſo join'd, to the ſecond Letter, one place forwards (either to the Left or Right-hand) it muſt needs follow,

That the *Unciæ* of the ſecond Term (in any ſuch Power) will always be the Sum of ſo many Units added together more one, as there hath been Multiplications of the firſt Root, which will always be determined by the Index of the firſt Term in the Power.

And, becauſe the *Unciæ* of all the intermediate Terms are only remov'd along with their Letters, it alſo follows, that if they are added together, their reſpective Sums will produce the true *Unciæ* of the intermediate Terms in the new rais'd Power; As doth plainly appear from the following Numbers ſo remov'd without their Letters; which both ſhews and Demonſtrates an eaſy Way of producing the *Unciæ* of any ordinary Power (*viz.* of one not very high) raiſed from either a Binomial, or Reſidual Root.

Thus

Add { 1 . 1 the *Unciæ* of the Root.
      1 . 1

Add { 1 . 2 . 1 the *Unciæ* of the Square.
      1 . 2 . 1

Add { 1 . 3 . 3 . 1 the *Unciæ* of the Cube.
      1 . 3 . 3 . 1

Add { 1 . 4 . 6 . 4 . 1 the *Unciæ* of the 4th Power.
      1 . 4 . 6 . 4 . 1

Add { 1 . 5 .10 .10 . 5 . 1 the *Unciæ* of the 5th Power.
      1 . 5 .10 .10 . 5 . 1

Add { 1 . 6 .15 .20 .15 . 6 . 1 the *Unciæ* of the 6th Power.
      1 . 6 .15 .20 .15 . 6 . 1

1 . 7 .21 .35 .35 .21 . 7 . 1 the *Unciæ* of the 7th Power.

And ſo on in this Manner *ad infinitum*.

Now

Now if these Numbers are prefixt to the aforesaid Letters, all the Terms will be compleated with their respective *Unciæ*, and will stand thus,

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

But that the Business of finding these *Unciæ*, may be render'd yet more easy for Practice, it will be convenient to consider what Series, or Progression the *Unciæ* of each Term do make, from the aforesaid Additions.

Unciæ of the First Term.	*Unciæ of the Second Term.*	*Unciæ of the Third Term.*	*Unciæ of the Fourth Term.*	*Unciæ of the Fifth Term.*	*Unciæ of the Sixth Term.*	*Unciæ of the Seventh Term.*	*Unciæ of the Eighth Term, &c.*	
1	1							*Unciæ of the single Quantities.*
1	2	1						*Unciæ of the Square.*
1	3	3	1					*Unciæ of the Cube.*
1	4	6	4	1				*Unciæ of the 4th Power.*
1	5	10	10	5	1			*Unciæ of the 5th Power.*
1	6	15	20	15	6	1		*Unciæ of the 6th Power.*
1	7	21	35	35	21	7	1	*Unciæ of the 7th Power.* &c.

The *Unciæ* of the first Terms, is only a Series of Units, whose Sum is every where the *Unciæ* of the second Term.

The *Unciæ* of the second Term, is a Series of Numbers in Arithmetick Progression; whose Sum is every where the *Unciæ* of the next superior Power in the third Term, and may be found by the Third Step. Chap. 1. Part 8.

That is in the seventh Power it will be $\frac{6+1}{2} \times 6 = 21$ the *Unciæ* of the third Term.

The rest of the *Unciæ* are a Compounded Series, whose respective Sums may be obtained from the *Unciæ* of their preceding Terms.

Thus $\frac{21 \times 5}{3} = 35$, then $\frac{35 \times 4}{4} = 35$, again $\frac{35 \times 3}{5} = 21$, also $\frac{21 \times 2}{6} = 7$, and $\frac{7 \times 1}{7} = 1$.

From hence may be deduc'd this general *Rule*.

Rule.

If the Index of the first Letter of any Term be Multiplyed into its own *Unciæ*, and that Product be Divided by the Number of Terms to that place; the Quotient will be the *Unciæ* of the next succeeding Term forward:

That is, by the help of those Indices that belong to the several Powers of the first or leading Letter only (as a) the true *Unciæ* of every Term may be easily found.

Example.

Let it be required to compleat all the Terms of the aforesaid several Powers; *viz.* $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$, with their proper *Unciæ*.

1. The Index of a^7 the first Term will be the *Unciæ* of the second Term. Thus $a^7 + 7a^6b$.

2. Then half the second Term's Index into its *Unciæ*, *viz.* $\frac{7 \times 6}{2} = 21$ will be the third Term's *Unciæ*.

Thus $a^7 + 7a^6b + 21a^5b^2$ will be the three first Terms.

3. Again $\frac{21 \times 5}{3} = 35$ is the *Unciæ* of the fourth Term.

Then it will be $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$.

4. And $\frac{35 \times 4}{4} = 35$ will be the *Unciæ* of the fifth Term.

Then $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$. &c. Until all the Terms are compleated with their respective *Unciæ*; And then they will stand

Thus, $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

Now here it may be further observed, that the *Unciæ* do only increase until the Indices of the two Letters become equal, or change places; and then the rest of the *Unciæ* will return or decrease in the same Order: That is, wherever the Indices of the Letters are alike, there the *Unciæ* will be alike.

And therefore it is not necessary to find the *Unciæ* (as before) beyond half the Number of Terms in any Power.

Corollary.

If m be equal to any whole Number; then the m^{th} Power of: $a \pm b$: will be $= a^{m} \pm m\,a^{m-1}\,b + m \times \frac{m-1}{2}\,a^{m-2}\,b^2 \pm m \times \frac{m-1}{2} \times \frac{m-2}{3}\,a^{m-3}\,b^3$, &c.

N. B.

N.B. The Sign $\pm$ denotes $+$ or $-$, and $\mp$ denotes $-$ or $+$.

Compound Quantities consisting of three or more Terms may be Involved to any Power required by Multiplication: But ought to be done here rather, by the Help of the same Power of a Binomial, in order to know the Nature and Composition of Powers, and thence the Method of Evolving them.

Thus; Suppose it was required to Involve $g+h-i$ to the third Power.

The third Power of the Binomial $a+b$ is $a^3+3aab+3abb+b^3$, which is your Canon for Involving.

$$\text{Then}\quad \begin{array}{l} g = a \text{ (i. e. 1st. } a\text{)} \\ h = b \text{ (i. e. 1st. } b\text{)} \\ \hline \therefore\ g^3 = a^3 \\ 3ggh = 3aab \\ 3ghh = 3abb \\ h^3 = b^3 \end{array}$$

$$\begin{array}{rrrrl} g^3 & +3ggh & +3ghh & +h^3 & = a^3 \\ & -3ggi & -6ghi & -3hhi & = 3aab \\ & & +3gii & +3hii & = 3abb \\ & & & -iii & = b^3 \end{array} \left\} \begin{array}{l} \text{Here } a \text{ (i. e. 2d. } a\text{)} \\ = g+h. \text{ And 2d. } b \\ = -i. \end{array} \right.$$

$$g^3 \begin{array}{l} +3h \\ -3i \end{array} gg \begin{array}{l} +3hh \\ -6hi \\ +3ii \end{array} g \begin{array}{l} +h^3 \\ -3hhi \\ +3hii \\ -iii \end{array}$$

Is the Cube of $g+h-i$, which was required.

CHAP. II.

Involution of Numbers.

ANY absolute Number being first reduc'd into its several Members (which are the several Significant Figures in the given absolute Number, with their due Number of Cyphers after each of them, and before such of them as are Decimals, along with the Point or Decimal Character) may be Involved to any required Power, by the help of the same Power of a Binomial (as in the foregoing Example in Species, which shews and demonstrates the manner of doing it); always observing to begin with the greatest Members of that Number.

Examples.

Examples.

1. Let it be required to find the Square of 5709.
First, The Square of the Binomial $a+b$ is $aa+2ab+bb$, which is your Canon for Involving.
Secondly, $5709 = 5000 + 700 + 9$.

Operation.

$$\begin{array}{rrl}
 & 5000 = a & \text{(i. e. 1st. } a) \\
 & 700 = b & \text{(i. e. 1st. } b) \\
\hline
(5000 \times 5000 =) & 25000000 = aa & \\
(2 \times 5000 \times 700 =) & 7000000 = 2ab & \\
(700 \times 700 =) & 490000 = bb & \\
\hline
(5700 \times 5700 =) & 32490000 = aa & \text{Here } a \text{ (i. e. 2d } a) \\
(2 \times 5700 \times 9 =) & 102600 = 2ab & = 5700. \text{ And } b \text{ (i. e.} \\
(9 \times 9 =) & 81 = bb & \text{2d. } b) = 9. \\
\hline
 & 32592681 & \text{is the Square required.}
\end{array}$$

2. What is the Cube of 463?
The Cube of the Binomial $a+b$ is $a^3 + 3aab + 3abb + b^3$. Canon.
$463 = 400 + 60 + 3$.

Operation.

$$\begin{array}{rl}
400 = a & \\
60 = b & \\
\hline
64000000 = a^3 & \\
28800000 = 3aab & \\
4320000 = 3abb & \\
216000 = b^3 & \\
\hline
97336000 = a^3 & \\
1904400 = 3aab & \text{Here } a = 460 \\
12420 = 3abb & \text{And } b = 3. \\
27 = b^3 & \\
\hline
99252847 & \text{the Cube of 463 Answer.}
\end{array}$$

Scholium.

The Square, Cube, &c. of any Integer Member will contain twice, thrice, &c. respectively so many Cyphers to the Right-hand as the said Member doth; wherefore in Involving, but more in Evolving Numbers, the following Table will be assisting.

Table.

The Root.	1	2	3	4	5	6	7	8	9
Square.	1	4	9	16	25	36	49	64	81
Cube.	1	8	27	64	125	216	343	512	729
Biquadrat.	1	16	81	256	625	1296	2401	4096	6561
5th Power.	1	32	243	1024	3125	7776	16807	32768	59049 &c.

3. What is the Square of 1.21?

1.21 is $= 1 + .2 + .01$

$$\begin{array}{r} 1 = aa \\ .4 = 2ab \\ .04 = bb \\ \hline 1.44 = aa \\ .024 = 2ab \\ .0001 = bb \\ \hline 1.4641 \text{ Answer.} \end{array}$$

$\left\{ \begin{array}{l} \text{Here } a = 1, \\ \text{and } b = .2. \end{array} \right.$

$\left\{ \begin{array}{l} \text{Here } a = 1.2 \\ \text{and } b = .01 \end{array} \right.$

Take one Example of mixt Involution of Numbers, the Converse of which is called the Numeral Exegesis.

5. If $y = 123$, what is the Value of $y^3 + y$?

Suppose $a + b = y$.

Then $a^3 + 3aab + 3abb + b^3 = y^3$.

And $a + b = y$.

Therefore $a^3 + 3a^2b + 3ab^2 + b^3 + a + b$ $\left\{ \text{Is the Canon for Involving.} \right.$

And $123 = 100 + 20 + 3$

Operation.

$$\begin{array}{r} 100 = a \\ 20 = b \\ \hline 1000000 = a^3 \\ 100 = a \\ 600000 = 3aab \\ 20 = b \\ 120000 = 3abb \\ 8000 = b^3 \\ \hline \end{array}$$

1728

$$\begin{array}{rl} 1728120 = & a^3 + a \\ 129600 = & 3aab \\ 3 = & b \\ 3240 = & 3abb \\ 27 = & b^3 \\ \hline 1860990 = & y^3 + y \quad \text{Answer.} \end{array}$$

Here $a = 120$
And $b = 3$.

Note, *That it is not usual to make use of the foregoing Cyphers; However, when they are not written, they are to be understood as written.*

CHAP. III.

Involution of Fractions.

THE *Rule* for Involving Fractional Quantities or Numbers, is this; *viz,*

Involve the Numerator into it self for a new Numerator, and the Denominator into it self for a new Denominator, each so often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b+d}{a-c}$
1 ⊖ 2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb+2bd+dd}{aa-2ac+cc}$
1 ⊖ 3	3	$\frac{bbb}{aaa}$	$\frac{27b^3c^3}{8a^3d^3}$	$\frac{b^3+3b^2d+3bd^2+d^3}{a^3-3a^2c+3ac^2-c^3}$

Again	1	$\frac{1}{463}$	$\frac{68}{5709}$
1 ⊖ 2	2	$\frac{1}{214369}$	$\frac{4624}{32592681}$

If x be equal to $\frac{132}{1000}$; then $2xx + 3x$ will be equal to $\frac{430848}{1000000}$.

PART. IV.

Evolution.

CHAP. I.

Evolution of whole Quantities.

EVolution is the Extracting of Roots from any given Power; that is, it is the converſe Work to that of Involution; and in ſingle Quantities it is eaſy, if the given Power have ſuch a Root as is required, which may be thus known.

If the given Power have no Numbers prefixt to it, and its Index can be Divided by the Number Denominating the Root required, the Quotient will be the Index of the Root ſought.

Thus, If the Cube-Root of *aaaaaa*, *viz.* a^6 were required: The Number that Denominates the Cube-Root is 3; then 3) 6 (2; that is $\sqrt[3]{a^6} = a^2$ the Root required. And ſuch Operations are uſually ſet down.

Thus	1	a^6	a^6b^6	$a^6b^6c^6$
1ɯ2	2	a^3	a^3b^3	$a^3b^3c^3$
1ɯ3	3	a^2	a^2b^2	$a^2b^2c^2$
3ɯ2	4	a	ab	abc

Note, *The Figures plac'd in the Margin after the Sign* (ɯ) *of Evolution, denote the Number Denominating the Root to be Extracted.*

If the given Powers have Co-efficients (*viz.* Numbers prefix'd to them) then you muſt Extract their reſpective Roots.

Thus	1	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
1ɯ2	2	$9a^2$	$36a^4b^4$	$144a^2b^2c^2$
1ɯ4	3	$3a$	$6a^2b^2$	$12abc$
2ɯ2	4	$3a$	$6aabb$	$12abc$

But

But if the Root required cannot be truly Extracted out of both Co-efficients and Indices of the given Power, then it is a Surd, and must have the Sign of the Root required prefix to it (or its Index Superior to it. See Part V.)

Thus	1	a^5	$49a^3$	$217b^3d^3$
1 ω 2	2	$\sqrt[2]{a^5}$	$\sqrt[2]{49a^3} = 7\sqrt{a^3}$	$\sqrt[2]{217b^3d^3}$
1 ω 3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{49a^3} = a\sqrt[3]{49}$	$\sqrt[3]{217b^3d^3} = bd\sqrt[3]{217}$

Evolution of Compound Quantities may be perform'd by the following

Rule.

First Involve any Binomial, as $x + y$, to such a Power as the Root to be Extracted requires; that is, to the second Power, if it be the Square-Root; to the third Power, if it be the Cube-Root; *&c.* that is to be Extracted. And the Power thus produc'd is your Canon for Evolving, which is to be used in the following Manner, *viz.*

First find the Root of the first, or greatest Term of the given Compound Quantity, and call it x, then Subtract that Term from the said Compound Quantity; the Remainder call your Resolvend; and the Value of the Co-efficient of y in the second Term of your Canon, call your Divisor: By this Divide the Resolvend, and the first Term of the Quotient is $= y$. *But here Note,* that in the Continuance of the Operation, you may make more of the first Terms of the Quotient $= y$. Then find the Sum of the Values of all the Terms of your Canon but the first; and call it your Ablativum, which Subtract from your Resolvend; and call the Remainder, if there be any, your (second) Resolvend. The Sum of the Values of the next foregoing x and y thus found, call x (*i. e.* your next x); and proceed with this Value of x, in order to find that of the next y as before is taught; and so on.

1. Suppose it was required to Extract the Square-Root of

$$a^4 \begin{matrix} -6n \\ +2z \end{matrix}\, a \begin{matrix} +9nn \\ -6nz \\ +zz \end{matrix}$$

1st. $: x + y : \times : x + y : = xx + 2xy + yy$, is your Canon for Extracting the Square-Root.

2dly. The Square-Root of aa (the first Term of the given Compound Quantity) is a; and therefore $a = x$ (*i. e.* 1*st.* x); and $a^2 = x^2$, which taken from the Compound Quantity, leaves $\begin{array}{l}-6n\\+2z\end{array}a\begin{array}{l}+9n^2\\-6nz\\+zz\end{array}$ for a Resolvend. The Co-efficient of y in the second Term of the Canon is $2x$, and its value $2a$ is your Divisor; by which the Resolvend being Divided, the first Term of the Quotient is $\begin{array}{l}-3n\\+z\end{array}$; wherefore $2xy + yy = \begin{array}{l}-6n\\+2z\end{array}a\begin{array}{l}+9nn\\-6nz\\+zz\end{array}$ is the Ablativum, which taken from the said Resolvend, leaves o: Consequently $a\begin{array}{l}-3n\\+z\end{array}$ is the Root sought.

Operation.

$$a^2\begin{array}{l}-6n\\+2z\end{array}a\begin{array}{l}+9nn\\-6nz\\+zz\end{array}\left(a\begin{array}{l}-3n\\+z\end{array}\right.\text{ Root.}$$

$$aa = xx$$

$$\begin{array}{l}-6n\\+2z\end{array}a\begin{array}{l}+9nn\\-6nz\\+zz\end{array}\text{ Resolvend.}$$

Divisor $2a = 2x$

$$\textit{Ablativum}\left\{\begin{array}{l}\begin{array}{l}-6n\\+2z\end{array}a = 2xy\\[2ex] \left.\begin{array}{l}+9nn\\-6nz\\+zz\end{array}\right\} = yy\end{array}\right. \qquad \text{Here } y = \begin{array}{l}-3n\\+z\end{array}$$

o *Remainder.*

Or the same Root may be Extracted in an easier, but more tedious Manner; thus,

$$a^2 - 6an + 9nn + 2az - 6nz + zz \ (a - 3n + z \text{ Root.}$$
$$a^2 = xx$$

$$-6an + 9nn + 2az - 6nz + zz \quad \textit{Resolvend.}$$

Divisor $2a = 2x$

Ablat.

Ablat. $\left\{\begin{array}{l} -6an = 2xy \\ +9nn = yy \end{array}\right.$ Here $y = -3n$.

$+2az - 6nz + zz$ *Resolvend.*

Divisor $2a - 6n = 2x$

Ablativum $\left\{\begin{array}{r} 2az - 6nz = 2xy \\ + zz = yy \end{array}\right.$ Here x (*i. e.* 2*d.* x) $= a - 3n$. And y (*i. e.* 2*d.* y) $= z$.

0 *Remainder.*

But if the propos'd Compound Quantity hath not such a Root as is required, the Evolution may be continued to an endless Series; thus,

2. Let it be required to Extract the Square-Root of $rr \pm zz$ nearly.

Operation.

$rr \pm zz \left(r \pm \frac{zz}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5} - \frac{5z^8}{128r^7}, \right.$ &c. *Sine Fine.*

$rr = xx$

$0 \pm zz$ *Resolvend.*

Div. $2r = 2x$

Ablat. $\left\{\begin{array}{l} \pm zz = 2xy \\ + \frac{z^4}{4rr} = yy \end{array}\right.$ Here $y = \pm \frac{zz}{2r}$

$- \frac{z^4}{4rr}$ *Resolvend.*

Divisor $2r \pm \frac{zz}{r} = 2x$

Here $x = r \pm \frac{zz}{2r}$

Ablat. $\left\{\begin{array}{l} - \frac{z^4}{4rr} \mp \frac{z^6}{8r^4} = 2xy \\ + \frac{z^8}{64r^6} = yy \end{array}\right.$ Here $y = - \frac{z^4}{8r^3}$

$\pm \frac{z^6}{8r^4} - \frac{z^8}{64r^6}$ *Resolvend.*

Divisor

Divisor $2r \pm \frac{zz}{r} - \frac{z^4}{4r^3} = 2x$ Here $x = r \pm \frac{zz}{2r} - \frac{z^4}{8r^3}$

Ablat. $\left\{ \begin{array}{r} \pm \frac{z^6}{8r^4} + \frac{z^8}{16r^6} \pm \frac{z^{10}}{64r^8} = 2xy \\ + \frac{z^{12}}{256r^{10}} = yy \end{array} \right.$ Here $y = \pm \frac{z^6}{16r^5}$

$-\frac{5z^8}{64r^6} \pm \frac{z^{10}}{64r^8} - \frac{z^{12}}{256r^{10}}$ *Resolvend.*

Divisor $2r \pm \frac{zz}{r} - \frac{z^4}{4r^3} \pm \frac{z^6}{8r^5} = 2x.$

Now, if by this Divisor, you Divide the next foregoing Resolvend, you'll find $-\frac{5z^8}{128r^7} \pm \frac{7z^{10}}{256r^9} - \frac{21z^{12}}{1024r^{11}} \pm \frac{33z^{14}}{2048r^{13}}$ for the Value of the next y, which added to $r \pm \frac{zz}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5}$, the Value of the next foregoing x, is nearly = the Square-Root of $rr \pm zz$.

Corollary.

Here you may see that the *Unciæ* of the Square-Root of a Binomial or Residual are equal to $1, \frac{1}{2}, \frac{1}{2} \times \frac{\frac{1}{2}-1}{2}, \frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \times \frac{\frac{1}{2}-2}{3}$; &c. that is, putting n = Index of the Square-Root, or $= \frac{1}{2}$, the *Unciæ* of the Square-Root of a Binomial or Residual are equal to $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. *In infinitum.*

3. Let it be required to Extract the Cube-Root of $r^3 \pm z^3$ nearly.

$:x+y:\times:x+y:\times:x+y: = x^3 + 3xxy + 3xyy + y^3$. Canon.

Operation.

Operation.

$$r^3 \pm z^3 \left(r \pm \frac{z^3}{3rr} - \frac{z^6}{9r^5}, \text{ \&c.} \right.$$
$$r^3 = x^3$$

$\pm z^3$ *Resolvend.*

Div. $3rr = 3xx$

$$\textit{Ablativum.} \begin{cases} \pm z^3 = 3xxy \\ + \dfrac{z^6}{3r^3} = 3xyy & \text{Here } y = \pm \dfrac{z^3}{3rr} \\ \pm \dfrac{z^9}{27r^6} = y^3 \end{cases}$$

$$- \frac{z^6}{3r^3} \mp \frac{z^9}{27r^6} \quad \textit{Resolvend.}$$

$$\textit{Divisor } 3rr \pm \frac{2z^3}{r} + \frac{z^6}{3r^4} = 3xx \quad \text{Here } x = r \pm \frac{z^3}{3rr}.$$

$$\textit{Ab.} \begin{cases} - \dfrac{z^6}{3r^3} \mp \dfrac{2z^9}{9r^6} - \dfrac{z^{12}}{27r^9} = 3xxy \\ + \dfrac{z^{12}}{27r^9} \pm \dfrac{z^{15}}{81r^{12}} = 3xyy \\ - \dfrac{z^{18}}{729r^{15}} = y^3 \end{cases} \begin{cases} \text{Here } y = - \dfrac{z^6}{9r^5}, \text{ or it} \\ \text{may be} = - \dfrac{z^6}{9r^5} \pm \\ \dfrac{5z^9}{81r^8} \end{cases}$$

$$\pm \frac{5z^9}{27r^6} * \mp \frac{z^{15}}{81r^{12}} + \frac{z^{18}}{729r^{15}} \quad \textit{Resolvend.}$$

$$\textit{Divisor } 3rr \pm \frac{2z^3}{r} - \frac{z^6}{3r^4} \mp \frac{2z^9}{9r^7} + \frac{z^{12}}{27r^{10}} = 3xx.$$

In this last Divisor x is $= r \pm \frac{z^3}{3rr} - \frac{z^6}{9r^5}$, by which Divisor, if you Divide its respective Resolvend, you'll have $\pm \frac{5z^9}{81r^8} - \frac{10z^{12}}{243r^{11}} \pm \frac{22z^{15}}{729r^{14}}$ for the Value of the next y. And therefore the Cube-Root of $r^3 \pm z^3$ is nearly $= r + \frac{z^3}{3rr} -$

$\frac{z^6}{9r^5} \pm \frac{5z^9}{81r^8} - \frac{10z^{12}}{243r^{11}} \pm \frac{22z^{15}}{729r^{14}}$.

Corollary.

Here you may see that the *Unciæ* of the Cube-Root of a Binomial or Residual are $1, \frac{1}{3}, \frac{1}{3} \times \frac{\frac{1}{3}-1}{2}, \frac{1}{3} \times \frac{\frac{1}{3}-1}{2} \times \frac{\frac{1}{3}-2}{3}$, *&c. In infinitum*; that is, putting $n =$ Index of the Cube-Root $= \frac{1}{3}$, the *Unciæ* of the Cube-Root of a Binomial or Residual are equal to $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, *&c. Sine Fine.*

Scholium.

In like Manner you may find that putting $n = \frac{1}{4}$, the *Unciæ* of the Biquadrat-Root of a Binomial or Residual are equal to $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, *&c. In infinitum*: And so on for superiour Roots.

From what has been said in Chap. 1. Part 3. and in the precedent *Corollaries* and *Scholium*, we have good Reason to believe that Universally the *Unciæ* of any Binomial or Residual, whose Index is n, are equal to $1, n, n \times \frac{n-1}{2}, n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, *&c*: But for a further Confirmation of this, I referr you to Part XV. Chap. 1.

CHAP. II, III.

Evolution, and mixt Evolution of Numbers.

NOte, *What I call mixt Evolution, is the Method of Extracting the Roots of adfected Equations.*

Rule.

When you are to Extract any unmixt Root; *viz.* the Square-Root, Cube-Root, *&c.* of a given Number; Involve any Binomial,

mial, as $x+y$ into it self according to the Number Denominating the Root to be Extracted; and the Power thus produc'd is your Canon for Evolving. But when you are to Extract any of the Roots of an Adfected Equation, suppose the Binomial $x+y=$ the Root you seek; then instead of the said Root (or unknown Letter) and its Powers in the Equation, substitute their respective Values, *viz.* $x+y$, and the respective Powers thereof; and the Sum of the Terms wherein either x or y occurs in the Equation thus had is your Canon for Evolving the absolute Number in this (or in the propos'd) Equation.

Having thus fram'd a Canon for Evolving, the Operation is to be perform'd in the following Manner.

1st. Find the first or greatest Member (*viz.* the first significant Figure, with its due Number of Cyphers) of the Root sought; and call it x; then having found the Value of the first Term, or of the Sum of the first Terms of your Canon, *i. e.* of all those Terms wherein x and its Powers only occur, Subtract it from the absolute Number; the Remainder call your Resolvend: And the Value of the Co-efficient, or of the Sum of the Co-efficients of y in the second Term or Terms of your Canon, call your Divisor. Now by Dividing the Resolvend by this Divisor the Value of y, or the second Member of the Root is found in some Cases, but not in all; wherefore, in the Beginning of your Operation, you must take Care that y be $=$ the greatest Member, and that the Sum of the Values of all the Terms of your Canon wherein y occurs, may not exceed the said Resolvend. Having thus found the Value of y, as also the Sum of the Values of all the Terms of your Canon wherein y occurs; place the former along with the before found Value of x in the Root, and call the latter your Ablativum, which Subtract from the said Resolvend; and, if there be a Remainder, call it your next Resolvend. The Sum of the Values of the foregoing x and y call x, (*i. e.* a 2*d.* x nearer the Truth.) And proceed with this Value of x, in order to find that of the next y as before is taught. And thus proceed 'till the Ablativum taken from its Resolvend leaves 0; or 'till you have as many Decimal Figures as you think sufficient.

Note, *Tho' by the first Division, you may not find the next Member of the Root sought; yet in continuing the Operation, one Division may serve to find several of the next following Members, or the Value of* y *to many Places of Figures, as will appear in the latter end of this* Part IV.

Examples.

1. Suppose it was required to Extract the Square-Root of 4624.

1st. $:x+y:\times:x+y:=xx+2xy+yy$ is your Canon for Extracting the Square-Root.

2dly. The * greatest Member of the Square-Root of 4624 is 60 (for $70\times70=4900$); therefore $x=60$, and $xx=3600$, which taken from 4624, leaves 1024 for a Resolvend; and the Co-efficient of y in the second Term of your Canon is $2x=120$ for your Divisor; by which Dividing the Resolvend, *viz.* 1024, the Quotient is $8=y$; therefore $2xy+yy=1024$ is the Ablativum, which taken from the Resolvend, leaves 0. Whence the Square-Root of 4624 is $60+8=68$.

* See the Table in Page 42.

Operation.

$$
\begin{array}{ll}
 & 4624\ (60+8=68 \text{ the Root required.} \\
 & \underline{3600=xx} \\
 & 1024\ \textit{Resolvend.} \\
\textit{Div.} & \underline{120=2x} \\
 & 960=2xy \\
 & \underline{64=yy} \quad \Big\}\ \text{Here } y=8 \\
 & \underline{1024\ \textit{Ablativum.}} \\
 & 0
\end{array}
$$

2. Let it be required to Extract the Cube-Root of 99252847.

$:x+y:\times:x+y:\times:x+y:=xxx+3xxy+3xyy+yyy$. Canon.

Operation.

$$
\begin{array}{l}
99252847\ (400+60+3=463 \text{ the Root required.} \\
\underline{64000000=x^3} \quad \Big\}\ \text{Here } x=400. \\
\underline{35252847\ \textit{Resol.}}
\end{array}
$$

See the Table in Page 42.

*Divisor	480000 = $3xx$		
	28800000 = $3xxy$		
	4320000 = $3xyy$	Here $y = 60$.	
	216000 = y^3		
	33336000 *Ablat.*		
	1916847 *Resol.*		
Divisor	634800 = $3xx$	Here $x = 460$.	
	1904400 = $3xxy$		
	12420 = $3xyy$	Here $y = 3$.	
	27 = y^3		
	1916847 *Ablat.*		
	0 *Remainder.*		

* *Here you may see that by Dividing the Resolvend by this Divisor, the Value of y thereby found, would be = 70; but the Ablativum so produc'd would exceed the Resolvend; wherefore y = 60.*

3. Suppose it was required to Extract the Biquadrat-Root of 66121111747853987761.

$:x+y:\times:x+y:\times:x+y:\times:x+y: = x^4 + 4x^3y + 6xxyy + 4xy^3 + y^4$. Canon.

Operation.

	66121111747853987761 (50000 + 700 + 9 = 50709 the (Root required.	
	62500000000000000000 = $xxxx$	
	3621111747853987761 *Resol.*	
Div.	500000000000000 = $4x^3$	
	3500000000000000000 = $4x^3y$	
	735000000000000000 = $6xxyy$	Here $y = 700$.
	6860000000000000 = $4xy^3$	
	240100000000 = $yyyy$	
	3574188401000000000 *Ablat.*	
	46929077753987761 *Resol.*	
Div.	521295372000000 = $4x^3$	Here $x = 50700$.
	4691658348000000 = $4x^3y$	
	1249258140000 = $6xxyy$	Here $y = 9$.
	147841200 = $4xy^3$	
	6561 = y^4	
	46929077753987761 *Ablat.*	
	0 *Remainder.*	

If

If the given Power hath not ſuch a Root as is required, you may notwithſtanding find a Root nearer the truth, than any aſſigned in the following manner.

Suppoſe it was required to find the Square-Root of 2 nearly.

Operation.

$2\ (1 + .4 + .01 + .004 + .0002 + \&c. =$

$(1.4142 +$ is the Square-Root of 2 nearly.

$1 = xx$

Diviſor $2 = 2x)\ 1$ *Reſol.*

$.8 = 2xy$ Here $y = .4$.

$.16 = yy$

$.96$ *Ablat.*

Div. $2.8 = 2x)\ .04$ *Reſol.*

$.028 = 2xy$ Here $x = 1.4$.

$.0001 = yy$ And $y = .01$.

$.0281$ *Ablat.*

Div. $2.82 = 2x)\ .0119$ *Reſol.*

$.01128 = 2xy$ Here $x = 1.41$.

$.000016 = yy$ And $y = .004$.

$.011296$ *Ablat.*

Div. $2.828 = 2x)\ .000604$ *Reſol.*

$.0005656 = 2xy$ Here $x = 1.414$.

$.00000004 = yy$ And $y = .0002$.

$.00056564$ *Ablat.*

$.00003836$ *Remainder.*

The common Method of Extracting the Square-Root, Cube-Root, &c. of Numbers, is only an Abridgement of the foregoing Method, and is thus perform'd, *viz.*

Place the firſt Point always over the Figure which is in the Place of the Units. Place alſo a Point over every other Figure, denoted by the Denominator of the Index of the Root to be Extracted; that is, if it be the Square-Root, Cube-Root, &c. that is to be Extracted, point over every *2d. 3d. &c.* Figures reſpectively to the Left-hand, and if there be any Decimals in the given Number, to the Right-hand of the Figure, which is

is in the Place of Units. And as many Points as there are over the Places of whole Numbers, ſo many Places of whole Numbers muſt be in the Root; and the reſt are Decimals.

Again, any Binomial, as $x + y$, being Involved according to the Denominator (1 being the Numerator) of the Index of the Root to be Extracted, produces your Canon for Evolving. Then,

1ſt. By the Table of Powers in *Page* 42, or otherwiſe, find the greateſt Power that is contained in the firſt Period towards the Left-hand; *viz.* the greateſt Square, if it be the Square-Root; the greateſt Cube, if it be the Cube-Root; *&c.* that is to be Extracted; then having plac'd the Root $= x$ in the place aſſigned for it, which is likewiſe call'd the Root; Subtract the ſaid Power from the ſaid Period.

After the Remainder place the Figures in the next Period, and call that Number your Reſolvend; call alſo the Value of the Co-efficient of y in the ſecond Term of your Canon your Diviſor. Then aſk how oft the Diviſor is contained in the Reſolvend, omitting all the Figures in the laſt mention'd Period but the firſt; the Anſwer or Number $= y$ ſet in the Root next after the Value of x. Then find the Ablativum thus: Place the Figures which are the Value of the ſecond Term of your Canon, ſo as the laſt of them may be under the firſt of the laſt mention'd Period, and the Values of the *3d*, *4th*, *5th*, *&c.* Terms one, two, three, *&c.* Places reſpectively, more to the Right-hand, than thoſe of the *2d.* Term; and the Sum of the *2d*, *3d*, *4th*, *5th*, *&c.* Terms, plac'd as aforeſaid, is the Ablativum, which take from the Reſolvend.

But here Note, that if the Ablativum thus found ſhould be greater than the Reſolvend, then the Value of y is too great, and muſt be made leſs.

After the Remainder place the Figures in the next Period, and call that Number your Reſolvend, and call the Figures plac'd in the Root x; by which find the Value of the next y, in like manner, as before directed; and ſo proceed 'till you have done with all your Periods: And if afterwards there is a Remainder, place Cyphers after it, in order to find as many Decimal Figures as you pleaſe.

Example.

Let it be required to Extract the Square-Root of *6968*, nearly.

Operation.

Operation.

6968 (83.4745
64 = xx

Divisor 16 = $2x$) 568 *Resolvend.*
48 = $2xy$
9 = yy

489 *Ablativum.*

Div. 166 = $2x$) 79.00 *Resolvend.*
664 = $2xy$
16 = yy

6656 *Ablativum.*

Div. 1668 = $2x$)124400 *Resolvend.*
116809 *Ablat.* = $2xy + yy$.

16694.4) 75910.0*
66777.6

166948.5) 913240.0
8347425

784975 *Remainder.*

* *Here I go on with the Operation as it ought to be practis'd.*

The other Figures of the Root to the 12th may be found by Division; thus,

166949.0) 784975.0 (47018
117179
0315
148

.15

Whence the Square-Root of 6968 is nearly = 83.474547018.

CHAP.

CHAP. III.

Evolution Mixt of Numbers, or the Method of Extracting the Roots of Affected Equations; which Method is generally call'd the Numeral Exegesis.

1. LET it be reqnired to find one of the Affirmative Values of a in this Equation, *viz.* $a^3 - 171.91aa + 7905.6a = 71460$.

Suppose $x + y = a$; then,

Canon.

$$\left.\begin{array}{rcl} aaa = & xxx + & 3xxy + 3xyy + y^3 \\ -171.91aa = & -171.91xx - & 343.82xy - 171.91yy \\ +7905.6a = & 7905.6x & +7905.6y \end{array}\right\} = 71460.$$

First, I suppose $a = 20$; then $a^3 - 171.91aa + 7905.6a$ $(= 8000 - 68764 + 158112) = 97348$, which is more than 71460; therefore $a \sqsupset 20$.

Again, I suppose $a = 10$; then $a^3 - 171.91aa + 7905.6a = 62865$ which is less than 71460; therefore $a \sqsubset 10$:

That is, a is $\sqsupset 20$, and $\sqsubset 10$; consequently 10 is the first Member of one of the Values of a; that is $10 = x$.

Operation.

$$71460\ (10 + 1 + .9 + .01 = 11.91 = a$$

$$\left.\begin{array}{rl} 1000 = & x^3 \\ -17191 = & -171.91xx \\ +79056 = & +7905.6x \end{array}\right\} \text{Here } x = 10$$

$$62865 = x^3 - 171.91xx + 7905.6x$$

8595 *Resolvend.*

$$\begin{array}{rl} 300 = & 3xx \\ -3438.2 = & -343.82x \\ +7905.6 = & 7905.6 \end{array}$$

4767.4 *Divisor.*

$300 = 3xxy$		
$-\ 3438.2 = -\ 343.82xy$		
$+\ 7905.6 = 7905.6y$	Here $y = 1$	
$+\ 30 = 3xyy$		
$-\ 171.91 = -\ 171.91yy$		
$+\ 1 = yyy$		
4626.49 *Ablativum.*		
3968.51 *Resolvend.*		
$363 = 3xx$		
$-\ 3782.02 = -\ 343.82x$	Here $x = 11.$	
$+\ 7905.6 = 7905.6$		
4486.58 *Divisor.*		
$326.7 = 3xxy$		
$-\ 3403.818 = -\ 343.82xy$	Here $y = .9$	
$+\ 7115.04 = 7905.6y$		
$+\ 26.73 = 3xyy$		
$-\ 139.2471 = -\ 171.91yy$		
$.729 = yyy$		
3926.1339 *Ablativum.*		
42.3761 *Resolvend.*		
$424.83 = 3xx$		
$-\ 4091.458 = -\ 343.82x$	Here $x = 11.9$	
$+\ 7905.6 = 7905.6$		
4238.972 *Divisor.*		
$+\ 4.2483 = 3xxy$	Here $y = .01$	
$-\ 40.91458 = -\ 343.82xy$		
$+\ 79.056 = 7905.6y$		
$+\ .00357 = 3xyy$		
$-\ .0171910 = -\ 171.91yy$		
$+\ .000001 = yyy.$		
42.3761 *Ablativum.*		
0 *Remainder.*		

Whence $a = 11.91$.
This Equation hath two Roots more, *viz.* 60 and 100.

2. If

2. If $a^3 + a$ be equal to 1860990, 'tis required to find one of the Values of a.

Suppose $x + y = a$, then

Canon.

$$\left.\begin{aligned} a^3 &= x^3 + 3xxy + 3xyy + y^3 \\ +\ a &= x + y \end{aligned}\right\} = 1860990$$

Operation.

1860990 (100 + 20 + 3 = 123 = a

1000000 = x^3	
100 = x	
1000100 = $x^3 + x$	
860890 *Resolvend.*	
30000 = $3xx$	
1 = 1	
30001 *Divisor.*	
600000 = $3xxy$	
20 = y	Here $y = 20$
120000 = $3xyy$	
8000 = y^3	
728020 *Ablativum.*	
132870 *Resolvend.*	
43200 = $3xx$	
1 = 1	Here $x = 100 + 20 = 120$.
43201 *Divisor.*	
129600 = $3xxy$	And $y = 3$.
3 = y	
3240 = $3xyy$	
27 = yyy	
132870 *Ablativum.*	
0 *Remainder.*	

Consequently 100 + 20 + 3 = 123 = a; that is, = one of the Values of a.

If

If any more of the Roots or Values of a, in the propos'd Equation be required; and that you can find the first Member ($= 1st. x$) of any such Root, then you may proceed to Extract that Root by the same Canon: But if you can't readily find any such first Member.

Reduce all the Terms of the Equation to one Side, making them equal to 0; next divide each part of this Equation, by $a -$ the known Root; afterwards transpose the Absolute Number in the Quotients; and then frame a Canon for this last Equation as hath been already taught, in order to Evolve the Absolute Number therein by that Canon.

But I find that $a^3 + a - 1860990 = 0$, Divided by $a - 123$ yields in the Quotients, $aa + 123a + 15130 = 0$; the Absolute Number in which being Transpos'd, will give this Quadratick Equation, $aa + 123a = -15130$, which Equation hath no Root, either Affirmative or Negative; wherefore no other Root of, or in the propos'd Equation can be found by this Method but 123; notwithstanding the above Quadratick, and consequently the propos'd Equation hath two Imaginary Roots, that is to say, Impossible Roots, which are found by Part X. to be $\frac{-123 - \sqrt{-45391}}{2}$ and $\frac{-123 + \sqrt{-45391}}{2}$, which along with 123 are the three Roots or Values of a in the propos'd Cubick Equation.

CHAP. IV.

Evolution of Fractions.

First prepare the Fractions propos'd to be Evolved, *viz.* by reducing Compound Fractions to simple ones, as also Mixt Fractions to Fractions of the same Denomination, and by reducing them to their least Terms, by *Sect. 4th, 1st*, and *3d.* Chap. II. Part. II Then

If the Numerator and Denominator have each of them such a Root as is required, Evolve them, and their respective Roots will be the Numerator and Denominator of the new Fraction required.

Ex-

Examples.

1. Thus the Square-Root of $\frac{1}{a}$ of $\frac{gg}{b}$ of $\frac{a}{b} = \frac{gg}{bb}$ is $= \frac{g}{b}$

2. Also the Cube-Root of $1 + \frac{6aab + 2bbb}{a^3 - 3aab + 3abb - b^3} = \frac{a^3 + 3aab + 3abb + b^3}{a^3 - 3aab + 3abb - b^3}$ will be found to be $\frac{a+b}{a-b}$.

3. The Cube-Root of $\frac{1296}{750}$, that is of $\frac{216}{125}$ is $\frac{6}{5}$.

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not, or the Denominator may have such a Root, when the Numerator hath not. In such Cases the Roots may be set down as follow; *viz.* The Square-Root of $\frac{900}{7cf}$ is $= \frac{30}{\sqrt{7cf}}$; and the Cube-Root of $\frac{40}{27c^3d^3}$ is $= \frac{\sqrt[3]{40}}{3cd}$.

But when neither the Numerator, nor the Denominator (being in their least Terms) have just such a Root as is required; prefix the Radical Sign of the Root to the Fraction; thus the Square-Root of $\frac{a}{b}$ is $\sqrt{\frac{a}{b}}$: And the Cube-Root of $\frac{9}{7}$ is $\sqrt[3]{\frac{9}{7}}$.

Again, If the Fractional Numbers given can be reduc'd to Decimals of the same Value, you may reduce them to such, and then Evolve.

Examples.

1. Thus the Cube-Root of $\frac{1296}{750}$ is found to be 1.2, by Reducing $\frac{1296}{750}$ to a Decimal of the same Value (which is done by Dividing the Numerator, with a sufficient Number of Decimal Cyphers plac'd after it by the Denominator 'till nothing remains), and then Extracting the Cube-Root of that Decimal Fraction, that is of 1.728.

2. Also if $3y^3 - 2yy + 144y = 71\frac{7}{8}$, that is $= 71.875$; then one of the Values of y is $.5$.

3. Again

3. Again, If $a^5 + \frac{3}{4}a^3 = 5157\frac{5}{8}$, 'tis required to find one of the Values of a.

First reduce the Fractions in this Equation to Decimals of the same Value, and the said Equation will become $a^5 + .75a^3 = 5157.625$.

Secondly, Suppose $x + y = a$; then,

$$\left.\begin{array}{l} a^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\ +.75a^3 = +.75x^3 + 2.25x^2y + 2.25xy^2 + .75y^3 \end{array}\right\} = 5157.625.$$

Operation.

$$5157.625 \; (5 + .5 = 5.5 = a \quad \textit{Answer.}$$

$$\left.\begin{array}{rl} 3125 & = x^5 \\ 93.75 & = .75x^3 \end{array}\right\} \text{ Here } x = 5.$$

$$\begin{array}{rl} \hline 3218.75 & = x^5 + .75x^3 \\ \hline 1938.875 & \textit{Resolvend.} \\ \hline 3125 & = 5x^4 \\ 56.25 & = 2.25xx \\ \hline 3181.25 & \textit{Divisor.} \\ \hline \end{array}$$

$$\left.\begin{array}{rl} 1562.5 & = 5x^4y \\ 28.125 & = 2.25x^2y \\ 312.5 & = 10x^3yy \\ 2.8125 & = 2.25xyy \\ 31.25 & = 10xxy^3 \\ .09375 & = .75y^3 \\ 1.5625 & = 5xy^4 \\ .03125 & = y^5 \end{array}\right\} \text{ Here } y = .5.$$

$$\begin{array}{rl} \hline 1938.875 & \textit{Ablativum.} \\ \hline 0 & \textit{Remainder.} \end{array}$$

But if the Fractions in the propos'd Equation cannot be reduc'd to Decimals of the same Value (you may notwithstanding reduce them near the Truth, and then proceed to Extraction; or) Reduce the Equation to a common Denominator, and then Multiply each Part of that Equation by the common Denominator, &c. thus,

1. If

1. If $a^3 - 6\frac{109}{144}\ a = 2\frac{31}{144}$, 'tis required to find one of the Values of a.

The foregoing Equation being reduc'd to a common Denominator is $\frac{144a^3 - 973a}{144} = \frac{319}{144}$.

And, by Multiplying each part by 144, we have $144a^3 - 973a = 319$.

Now suppose $a = x + y$; then

$$\left.\begin{array}{l} 144a^3 = 144x^3 + 432xxy + 432xyy + 144y^3 \\ -973a = -973x \quad -973y \end{array}\right\} = 319$$

Operation.

$319\ (2 + .7 + .05 = 2.75 = a$ *Answer.*

$1152 = 144x^3$	
$-1946 = -973x$	
$-794 = 144x^3 - 973x$	
1113 *Resolvend.*	
$1728 = 432xx$	
$-973 = -973$	
755 *Divisor.*	
$1209.6 = 432xxy$	
$-681.1 = -973y$	
$423.36 = 432xyy$	Here $y = .7$.
$49.392 = 144y^3$	
1001.252 *Ablat.*	
111.748 *Resol.*	
$3149.28 = 432xx$	
$-973 = -973$	
2176.28 *Divisor.*	Here $x = 2.7$
$157.464 = 432xxy$	
$-48.65 = -973y$	Here $y = .05$
$2.916 = 432xyy$	
$.018 = 144y^3$	
111.748 *Ablat.*	
0 *Remainder.*	

Here you may observe that by Supposing $a = 2$, $144a^3 - 973a$ *is* = *a Negative Number; and consequently the Resolvend is greater than the Absolute Number 319: But if you had suppos'd* $a = 3$, *then* $144a^3 - 973a$ *wou'd be* $\sqsubset$ *said Absolute Number; And therefore* x *must not be suppos'd* = 3.

2. And,

2. And, If $a^4 + a = 126\frac{64}{81}$, then $81a^4 + 81a = 10270$; and one of the Values of a is $3.3333 + = 3\frac{1}{3}$.

N.B. *Tho' I have not hitherto in this* Part IV. *in Dividing any Resolvend by its respective Divisor, pursued such Division beyond finding the first Significant Figure of the Quotient, or of the Value of* y; *yet, after the Divisor once takes place, it may be continued to as many, or almost as many Figures, as the next preceding* x *hath of the first Figures of the Root sought; as in the following Examples.*

Example 1.

If $a^3 = 231$; Quere a *proxime.*
Suppose $x + y = a$; then
$(a^3 =) x^3 + 3xxy + 3xyy + y^3 = 231.$

231 (6
$216 = x^3$

15 *Resolvend.*

Divisor $108 = 3xx$

$14.04 = 3xxy$
$.3042 = 3xyy$ } Here $y = .13$.
$.002197 = y^3$

14.346397 *Ablativum.*

.653603 *Resolvend.*

Divisor $112.7307 = 3xx$ Here $x = 6.13$.

$.6527107533 = 3xxy$
$6165082 - = 3xyy$ } And $y = .00579$.
$1941 + = yyy$

.6533274553 + *Ablat.*

.0002755447 − *Resol.*

Divisor $112.9437 + = 3xx$. Here $x = 6.13579$, and the (next $y = .00000243966$;

Wherefore a is nearly $= 6.13579243966$.

Example

Example 2.

If $a^4 - 10a = 10000$; Quere *a proxime.*
Suppose $x + y = a$; then

$$\left.\begin{array}{l}(a^4 =) \quad x^4 + 4x^3y + 6xxyy + 4xy^3 + y^4 \\ (-10a =) -10x - 10y\end{array}\right\} = 10000$$

Operation.

$$\begin{array}{rl}
 & 10000 \ (10 \\
 & 10000 = x^4 \\
- & 100 = -10x \\ \hline
 & 9900 = x^4 - 10x \\ \hline
 & 100 \ \textit{Resolvend.} \\ \hline
 & 4000 = 4x^3 \\
- & 10 = -10 \\ \hline
 & 3990 \ \textit{Divisor.} \\ \hline
 & 100 = 4x^3y \\
- & .25 = -10y \\
 & .375 = 6xxyy \\
 & ....625 = 4xy^3 \\
 & ....39+ = y^4 \\ \hline
 & 100.12562539+ \ \textit{Ablativum.} \\ \hline
- & .12562539+ \ \textit{Resolvend.} \\ \hline
 & 4030.075 + = 4x^3 \\
- & 10 = -10 \\ \hline
 & 4020.075 + \ \textit{Divisor.}
\end{array}$$

Here $y = .025$

Here $x = 10.025$; And the next $y = -.000031249$;

Therefore a is nearly $= 10.024968750$.

Scholium.

The Method us'd in these two last Examples is more expeditious than that in the former Examples: But from this a better Method (which is the same with Mr. *Raphson*'s, and by him call'd the Converging Series) may be deduced; thus in

 Extracting

Extracting the Cube-Root of 231, in *Ex.* 1.) You may readily see that each Divisor is $=3xx$, each Resolvend $= 231 - x^3$, and consequently, from the Time the Divisor takes place, or serves to discover the first

This is the most natural way of raising Raphson's *Theorems for Extracting the Roots of Equations.*

Figure of the true Value of y, each $y = \frac{231 - xxx}{3xx}$ nearly; and therefore the Theorem for Extracting the Cube-Root of any Number $=$ A is $y = \frac{A - x^3}{3xx} \ldots$ In like Manner, in *Ex.* 2. *viz.* $a^4 ** - 10a = 10000$, each Divisor is $= 4x^3 - 10$, each Resolvend $= 10000 - x^4 + 10x$; And, from the Time the Divisor takes place, each $y = \frac{10000 - x^4 + 10x}{4x^3 - 10}$ nearly; And therefore the Theorem for finding the Value of a in this Equation, *viz.* $a^4 - pa = q$, is $y = \frac{q - x^4 + px}{4x^3 - p} \ldots$. And in this, or the like Manner, Theorems may be rais'd for Extracting any, or all the Affirmative, or * Negative Root, or Roots of any Equation.

* *See the 1st. Art. next after the 4th Parag. in* Chap. II. Part IX.

Wherefore in Extracting any Root of any Equation, whether Simple or Adfected, Let x be taken $=$ the Value of the first Figure, or *1st*, and *2d* Figures of the Root sought; then, by its respective Theorem, find the Value of the *1st* y to so many Figures as you think to be the true ones; the Sum of which and that of x, call x (*i. e.* your next x), with which proceed, by its respective Theorem, to find the Value of the next y; and so on.

Thus; if the Cube-Root of 231 is to be Extracted by this Method; Let *1st.* x be taken $= 6$; then *1st.* y $\left(= \frac{A - x^3}{3xx} \ldots = \frac{231 - 216}{108} \ldots\right) = .13$; therefore $6 + .13 = 6.13$ ($=$ *1st.* x $+$ *1st.* y) $=$ *2d.* x; then *2d.* y $\left(= \frac{231 - 230.346397}{112.7307} \ldots\right) = .00579$; therefore 6.13579 ($=$ *2d.* x $+$ *2d.* y) $=$ *3d.* x.

Then *3d.* y $\left(= \frac{231 - 230.9997244553+}{112.9437+} \ldots\right) = .00000243966$; therefore 6.13579243966 ($=$ *3d.* x $+$ *3d* y) is nearly $=$ the Cube-Root of 231.

You

You may see by this Example, and by as many more as you are pleas'd to try, that, from the Time the Divisor takes place, each Renewal doubles intirely, or almost the true Figures in the last x; *and, of consequence, a few Renewals afterwards will serve to Extract any Surd-Root required to very many places of Figures.*

PART. V.

Of the Indices, or Exponents, of Powers.

IF an Unit be Multiplyed by any Quantity a, and the Product a by a, and that Product aa by a, and that Product aaa by a, &c; the several Products are the 1*st*, 2*d*, 3*d*, 4*th*, &c. Powers of a.

Formerly the Manner of Writing the Root and the several Powers of a was as follows.

	Root or 1*st*. Power,	Square or 2*d*. Power,	Cube or 3*d*. Power.
Thus	a ,	aa ,	aaa ,
Or thus	a ,	aq ,	ac ,

Biquadrat or 4*th* Power
$aaaa$ &c. of a.
aqq

But, of late, the Powers of a are more usually designed in the following Manner, *viz.*

Root,	Square,	Cube,	4*th* Power,	&c. of a.
a or a^1 ,	a^2 ,	a^3 ,	a^4 ,	&c.

The Figures 1, 2, 3, 4, &c. writ Superior to a, and shewing its Powers, are called Indices or Exponents.

It is evident that the Powers of a in (or the Terms of) the foregoing Series, are in a continual Geometrical Proportion, whose Ratio is the 1*st*. Power or Root a, and the Indices of those Powers in a continued Arithmetical Proportion, whose common Excess is 1, by the Definitions of both Proportions.

Now,

Now, since the Exponent of each Power of the foregoing Series is, by the common Excess of the Indices, to wit by 1, more than the Index of the next foregoing Power, it must follow of Course, if the Series be continued backward, that the several Terms of it will be found (by Subtracting from the Index 1 of the Root a^1 the common Excess of the Indices, to wit 1, and from the Remainder 0 the said common Excess, and from this Remainder -1, the said common Excess, and from this Remainder -2 the said common Excess, &c.) $=$ &c. a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, a^4, &c.

And, since each Power of the said foregoing Series is the Product of the next foregoing Power and of the common Ratio; of Consequence, if that Series be continued backward, the several Terms of it will be found (by Dividing the Root a by the Ratio a, and the Quotient 1 by the said Ratio, and this Quotient $\frac{1}{a}$ by the said Ratio, and this Quotient $\frac{1}{a^2}$ by the said Ratio, &c.) $=$ &c. $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, &c.

From the two last Paragraphs, it is manifest that a^0 is $=1$, also $a^{-1}=\frac{1}{a}$, also $a^{-2}=\frac{1}{a^2}$, also $a^{-3}=\frac{1}{a^3}$, &c. And that the Series being continued backward and forward, may be writ

Thus, &c. a^{-5}, a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, a^4, a^5 &c.

Or thus, &c. $\frac{1}{a^5}$, $\frac{1}{a^4}$, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, a^5, &c.

From what hath been said, may be deduced and demonstrated the two Fundamental Rules for all Operations relating to the Exponents of Powers.

The 1st. of which is, that the Exponent of the Product of any two Terms of a Series in $\div\!\!\div$ (*viz.* of such a Series as the last but one) is equal to the Sum of the Exponents of those two Terms.

And the 2d. is, that the Exponent of the Quotient of any two Terms in a Series in $\div\!\!\div$ is had by Subtracting the Exponent of the Divisor from the Dividend's Exponent.

The first Rule may be prov'd by Algebraical Multiplication, by the help of what hath been already said in this Part V. So the Exponent of the Product of a^6 and a^4 is prov'd to be $6+4=10$, by Multiplying a^6 by a^4; for $a^6\times a^4$ is $=a^{10}$ $=a^{6+4}$. In like manner the Exponent of the Product of a^{-4} and a^3 may be proved to be $=-4+3=-1$; thus a^{-4} is

is (by what is abovesaid) $=\frac{1}{a^4}$, and $\frac{1}{a^4}\times a^3=\frac{a^3}{a^4}=\frac{a^3}{a^3}$ $\div a=\frac{1}{a}$; but $\frac{1}{a}$ is $=a^{-1}$; consequently -1 is the Exponent of $a^{-4}\times a^3$.

And the second Rule must follow of course from the First, or may be proved by Algebraical Division; so the Exponent of $a^6\div a^4$ is prov'd to be $6-4=2$, by Dividing a^6 by a^4; for $a^6\div a^4=\frac{a^6}{a^4}=a^2$. Also the Exponent of the Quotient of a^3 Divided by a^5 is prov'd to be $3-5=-2$; thus $\frac{a^3}{a^5}$ $=\frac{a^3}{a^3}\div a^2=\frac{1}{a^2}=a^{-2}$ (by what has been prov'd in this Part V.) In like manner the Exponent of the Quotient of a^{-4} Divided by a^{-3} is prov'd to be $=-4+3=-1$; thus a^{-4} (by what has been said in this Part) is $=\frac{1}{a^4}$, and $a^{-3}=\frac{1}{a^3}$; but $\frac{1}{a^4}\div\frac{1}{a^3}$ is $=\frac{a^3}{a^4}=\frac{a^3}{a^3}\div a=\frac{1}{a}$, that is $=a^{-1}$. &c.

The Method of Raising any Quantity to any required Power, or Extracting any Root out of it, is easily deduc'd from the two foregoing Rules; thus the Square of $a^3=a^3\times a^3$ is $=a^{3+3}$ $=a^6$, and $6=3\times 2$; that is, the Exponent of the Square is double the Exponent of the Root; or if you suppose a^6 to be the Root, then a^3 will be the Square-Root, wherefore since $3\div 6=\frac{1}{2}$) you may conclude that the Exponent of the Square-Root is half the Index of the Root. Also the Cube of $y^3=$ $y^3\times y^3\times y^3=y^{3+3+3}=y^9$, and $9=3\times 3$, which shews that the Exponent of the Cube is treble the Exponent of the Root; or that the Exponent of the Cube-Root is one third Part of the Exponent of the Root. Also $y^{-1}\times y^{-1}\times y^{-1}$ is $=$ $y^{-1-1-1}=y^{-3}$. And Universally the Exponent of the m-Power is m times the Exponent of the Root; and the Exponent of the m-Root (or $\frac{1}{m}$-Power) is $\frac{1}{m}$ Times the Exponent of the Root.

Hence the m Power of the m Root is $=$ the Root; that is the Square of the Square-Root, or the Cube of the Cube-Root, &c. is equal to the Root.

By the last Paragraph but one, it is evident that $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c. are equal to $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, &c. respectively; for a^1 being the Root, half its Index, to wit, the half of 1 must be the Index of the Square-Root; also $\frac{1}{3}$ of the Index of the Root must be the Index of the Cube-Root; &c.

☞ The Index of the Square-Root, Cube-Root, &c. may be otherwise discovered, thus;

Let it be required to find the Square-Root a^3.
Suppose it (to wit the Square-Root of a^3) $= a^u$.
Then $a^u \times a^u = a^3$; that is the Square of the Square-Root is equal to the Root; wherefore (by the first Rule) $u + u = 3$, and $u = \frac{3}{2}$; consequently $a^{\frac{3}{2}}$ is the Square-Root of a^3.
Again let it be required to find the Cube-Root of a^{-2}.
Suppose it $= a^w$; then $a^w \times a^w \times a^w = a^{3w}$ (by the 1*st* Rule) is $= a^{-2}$; wherefore $3w = -2$, and $w = -\frac{2}{3}$; consequently $a^{-\frac{2}{3}}$ is the Cube-Root of a^{-2}. &c.
So that this Remark is only a Confirmation of the above mentioned Paragraph.
The Sum of what hath been said in relation to the Indices or Exponents of Powers is, putting a^m and a^n equal to any two Quantities, that

1*st*. $a^m \times a^n = a^{m+n}$

2*d*. $a^m \div a^n = a^{m-n}$

3*d*. a^m rais'd to the n^{th} Power $= a^{mn}$

4*th*. The n-Root of $a^m = a^{\frac{m}{n}}$

Here follow more Varieties, which, in reality, are Included in the four last Articles, since m and n are Universal, and consequently may be equal to any Quantities Whole or Fracted, Affirmative or Negative.

$a^{-q} \times a^{+p} = a^{-q+p}$
$a^{-q} \div a^p = a^{-q-p}$
$a^p \div a^{-q} = a^{p+q}$
a^{-q} rais'd to p Power $= a^{-qp}$
$a^{-q} \times a^{-r} = a^{-q-r}$
$a^{-q} \div a^{-r} = a^{-q+r}$
the r-Root of a^{-q} is $= a^{\frac{-q}{r}}$

PART VI.

Of Surd-Roots.

CHAP. I.

Definitions.

1. WHen any Number or Quantity hath its Root propos'd to be Extracted, and yet is not a true figurate Number of that Kind; that is, if its Square-Root being demanded, it ſelf is not a true Square; if its Cube-Root being required, it ſelf be not a true Cube; &c. that Root is called a Surd-Root, becauſe it can never be exactly Extracted; and ſuch Roots are uſually deſign'd with their Indices, or Radical Signs: So the Square-Root of 2 is writ thus $2^{\frac{1}{2}}$, or thus $\sqrt{2}$; for 'tis evident that the Square-Root of 2 is not a whole Number; neither is it a Fraction, becauſe the Square of any Fraction is alſo a Fraction (by 16. 8. *Eucl. El.*); and conſequently it is not expreſſible by any Rational Number. Alſo the Cube-Root of b^2 is writ thus $\overline{b^2}|^{\frac{1}{3}}$ $(= b^{\frac{2}{3}})$, or thus $\sqrt[3]{b^2}$, &c.

2. But, altho' theſe Surd-Roots (when truly ſuch) are inexpreſſible by Rational Numbers, they are notwithſtanding capable of Algebraical Operations.

3. Surds are either Simple, which are expreſs'd by one ſingle Term as $b^{\frac{1}{2}}$, $\sqrt[3]{c}$, $\overline{bd}|^{\frac{1}{4}}$, &c. or Compound, which are form'd by the Addition and Subtraction of Simple Surds, as $\sqrt{5}+\sqrt{2}$, $5^{\frac{1}{2}} - 2^{\frac{1}{2}}$, $d^{\frac{1}{2}} + b^{\frac{1}{3}} - c^{\frac{1}{4}}$; &c. or elſe Univerſal as $\overline{7+2^{\frac{1}{2}}}|^{\frac{1}{3}}$ which ſignifies the Cubick-Root of that Number, which is the Reſult of adding 7 to the Square-Root of 2; &c.

4. Surds are alſo Commenſurable or Incommenſurable.

Commenſurable Surd-Roots are ſuch, whoſe Ratio or Proportion to one another, may be expreſs'd by Rational Numbers or Quantities:

And ſuch Surd-Roots whoſe Ratio cannot be expreſs'd by Rational Numbers or Quantities are call'd Incommenſurable.

CHAP.

CHAP. II.

Reduction of Simple Surds.

Proposition 1.

TO reduce Simple Heterogeneal Surds, that is, such as are under different Indices, to the least Homogeneals, or to such as shall have the least common Index.

Rule.

Reduce the Indices of the Surds to a common Denominator; next place the greatest common Measure of the Numerators as a Numerator over the common Denominator, which Fraction, when reduc'd to its lowest Terms, is the common Index sought: Then divide each of the said Indices by the common Index; aud then Involve each propos'd Number or Quantity without its Index, according to the Index denoted by its respective Quotient, and Superior to each Power, thus produc'd, place the said common Index, and you will have the Surds required.

Examples.

1. Reduce $\overline{12}|^{\frac{1}{4}}$ and $\overline{7}|^{\frac{1}{6}}$ to two other Surds of the same Value that will have a common Index.

$\frac{1}{4}$ and $\frac{1}{6}$ reduc'd to a common Denominator, are equal to $\frac{6}{24}$ and $\frac{4}{24}$. The greatest common Measure of the Numerators 6 and 4 is 2: Wherefore $\frac{2}{24}$, when reduc'd to its last Terms, is $\frac{1}{12}$ the common Index sought; by which the 1*st.* ($\frac{1}{4}$) and 2*d.* ($\frac{1}{6}$) Indices being Divided, the Quotients are 3 and 2 respectively.

The first of the propos'd Numbers without its Index, *viz.* 12 Involved according to the Index (3) denoted by the first Quotient, is $12^3 = 1728$.

Also the second propos'd Number without its Index, to wit 7 Involved according to the Index (2) denoted by the second Quotient, is $7^2 = 49$.

Consequently $\overline{1728}|^{\frac{1}{12}}$ and $\overline{49}|^{\frac{1}{12}}$ are the Surds required respectively.

2. Reduce $b^{\frac{2}{3}}$ and $\overline{dc}|^{\frac{2}{9}}$ to two other Surds of the same Value that shall have a common Index.

$\frac{2}{3}$ and $\frac{2}{9}$ reduc'd to a common Denominator are equal to $\frac{18}{27}$ and $\frac{6}{27}$ respectively.

The greatest common Measure of the Numerators is 6.

Wherefore $\frac{2}{9}$ ($= \frac{6}{27}$) is the common Index sought,

By which the 1*st.* ($\frac{2}{3}$) and 2*d.* ($\frac{2}{9}$) Indices being divided, the Quotients will be 3 and 1 respectively.

The first Quantity without its Index, to wit b Involved according to the Index (3) denoted by the first Quotient is b^3.

Also the second Quantity without its Index, that is dc Involved according to the Index (1) denoted by the second Quotient is cd.

Wherefore $\overline{b^3}|^{\frac{2}{9}}$ and $\overline{cd}|^{\frac{2}{9}}$ are the Surds required.

3. Let it be required to reduce $\overline{a}|^{\frac{i}{n}}$ and $\overline{e}|^{\frac{t}{m}}$ to two other Surds of the same Value that shall have a common Index.

$\frac{i}{n}$ and $\frac{t}{m}$ reduc'd to a common Denominator are equal to $\frac{mi}{mn}$ and $\frac{nt}{mn}$ respectively; and the greatest (known) common Measure of the Numerators mi and nt is 1; wherefore (by our Rule) $\frac{1}{mn}$ is the common Index sought; By which the 1*st.* $\left(\frac{i}{n}\right)$ and 2*d.* $\left(\frac{t}{m}\right)$ Indices being divided, the 1*st* and 2*d* Quotients are mi and nt respectively. The 1*st.* propos'd Quantity without its Index, that is a, Involved according to the Index (mi) denoted by the 1*st.* Quotient is a^{mi}: And the 2*d.* propos'd Quantity without its Index, *viz.* e, Involved according to the Index (nt) denoted by the 2*d.* Quotient is e^{nt}; wherefore (by our Rule) $\overline{a^{mi}}|^{\frac{1}{mn}}$ and $\overline{e^{nt}}|^{\frac{1}{mn}}$ are the Surds required.

Here you have three Examples at large of our Rule, the last of which, being Universal, may serve to Demonstrate it.

For $\overline{a^{mi}}|^{\frac{1}{mn}}$ is $= a^{\frac{mi}{mn}}$, which is manifestly $= a^{\frac{i}{n}}$.

And $\overline{e^{nt}}|^{\frac{1}{mn}}$ is $= e^{\frac{nt}{mn}}$, which is manifestly $= e^{\frac{t}{m}}$.

Corollary.

Hence Rational Quantities may be reduc'd to the Form of Surds, which shall have any assigned Indices. Thus,

Thus; If it was required to reduce a to the Form of a Surd, whose Index may be $\frac{1}{n}$.

First Divide the Index of a, to wit 1, by $\frac{1}{n}$, and the Quotient is n: Then a Involv'd according to the Index n gives a^n; Consequently $\overline{a^n}|^{\frac{1}{n}}$ is the Quantity required, which manifestly is $=a$,

Prop. 2.

To reduce Surd-Roots to their most simple Terms.

Rule.

First find the greatest Power, which, being affected with the Index (or Radical Sign) of the propos'd Surd, shall have a Rational Root, and will also measure the propos'd Number or Quantity without its Index: By which Power Divide the said Number or Quantity without its Index; and to the Quotient affected with the said Index, prefix the Rational Root with the Sign $\times$: So you will have a new Surd equal to the propos'd one, and in more Simple Terms.

☞ But when such a Power, by which the Division necessary to such Contraction is to be perform'd, cannot be readily discern'd; search out all the Numbers or Quantities (by Lemma to Chap. V. Part XII.) which will Divide the propos'd Number or Quantity, omitting its Index or Radical Sign, without a Remainder; and the greatest of those Divisors, which, being affected with the Index of the propos'd Surd, hath a Rational Root, is the greatest Power by which the Division, in the Manner aforesaid, is to be perform'd.

Examples.

1. So instead of $\sqrt{63}$, you may write $3\sqrt{7}$; for 9 is the greatest Square that can be had in 63 which will divide it without a Remainder: And $63 \div 9 = 7$; wherefore (by our Rule) $\sqrt{63}$, in its simplest Terms, is $3 \times \overline{7}|^{\frac{1}{2}} = 3\sqrt{7}$.

2. Also, if it was required to reduce $\overline{a^3b + aabb}|^{\frac{1}{2}}$ to its simplest Terms, it may be done thus; *viz.*

a^2 is the greatest Power, which being affected with the Index $\frac{1}{2}$ will have a Rational Root, and will also measure $a^3b + a^2b^2$.

And: $a^3b + a^2b^2 : \div a^2 = ab + bb$; consequently $\overline{a^3b + a^2b^2}|^{\frac{1}{2}}$, in its simplest Terms, is $a \times \overline{ab + bb}|^{\frac{1}{2}} = a\sqrt{:ab + bb}$:

3. And Universally $\overline{a^n b}|^{\frac{1}{n}}$ reduc'd to its simplest Terms is $a \times \overline{b}|^{\frac{1}{n}}$:

For a^n is the greatest Power which, being adfected with the Index $\frac{1}{n}$, will have a Rational Root, and by which $a^n b$ being also Divided, will leave no Remainder: And $a^n b \div a^n = b$; wherefore (by our Rule) $\overline{a^n b}|^{\frac{1}{n}}$, being reduc'd to its most simple Terms, is $a \times b^{\frac{1}{n}} = a \sqrt[n]{b}$.

This last Example is an Universal one, by which the foregoing Rule may be demonstrated thus.

$\overline{a^n}|^{\frac{1}{n}}$ is $= a$; and $\overline{a^n b}|^{\frac{1}{n}} = \overline{a^n}|^{\frac{1}{n}} \times \overline{b}|^{\frac{1}{n}}$: Wherefore $\overline{a^n b}|^{\frac{1}{n}} = a \times b^{\frac{1}{n}}$.

And that $a \times \overline{b}|^{\frac{1}{n}}$ is the least Term to which $\overline{a^n b}|^{\frac{1}{n}}$ can be reduc'd is certain, if a^n be the greatest n Power which can be had in $a^n b$. *Q. E. D.*

Prop. 3.

To find whether two Surd-Roots propos'd are commensurable or not.

Rule.

If they have different Indices, reduce them to their least common Index (by the 1*st*. Rule:) And then, if they are Fractions, reduce them to a common Denominator (by *Sect*. 2. Chap. II. Part II.) Then

Divide (or Multiply) them severally by such a Number or Quantity as will give one Quotient (or Product) Rational; then if the other Quotient (or Product) be Rational too, the Surds are commensurable, and in Proportion to each other as the Quotients or Products; but, if the other Quotient (or Product) be not Rational, the Surds are Incommensurable.

N.B. *The Divisor commonly us'd in this Case, is the greatest common one, which will give Rational Quotients, if the Surds be Commensurable.*

Ex-

Examples.

1. If it be required to find whether $\sqrt{12}$ and $\sqrt{3}$ are commensurable, and in what Proportion to one another, it may be done thus;

The greatest common Measure of $\sqrt{12}$ and $\sqrt{3}$ is $\sqrt{3}$, by which $\sqrt{12}$ and $\sqrt{3}$ being Divided, the respective Quotients are $\sqrt{4}$ and $\sqrt{1}$, that is 2 and 1, wherefore $\sqrt{12}$ and $\sqrt{3}$ are (by our Rule) commensurable, and the former is in Proportion to the latter as 2 to 1; *viz.* $\sqrt{12} \cdot\cdot \sqrt{3} :: 2 \cdot\cdot 1$.

2. Let it be required to find whether $\overline{\frac{b}{c}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}\Big|^{\frac{3}{2}}$ are Commensurable, and in what Proportion they are to one another.

$\overline{\frac{b}{c}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}\Big|^{\frac{3}{2}}$ being reduc'd to their least common Index are $\overline{\frac{b}{c}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c^3}{b^3}}\Big|^{\frac{1}{2}}$; and these reduc'd to a common Denominator, are equal to $\overline{\frac{b^4}{cb^3}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c^4}{cb^3}}\Big|^{\frac{1}{2}}$, which being Divided by their greatest common Divisor $\overline{\frac{1.}{cb^3}}\Big|^{\frac{1}{2}}$, the Quotients are Rational, and equal to b^2 and c^2.

Consequently the two Surds are Commensurable to one another, and in the Ratio of b^2 to c^2; that is $\overline{\frac{b}{c}}\Big|^{\frac{1}{2}} \cdot\cdot \overline{\frac{c}{b}}\Big|^{\frac{3}{2}} :: bb :: cc$.

This may be demonstrated by Equating the Product of the Extreams to that of the Means; thus,

$\overline{\frac{b}{c}}\Big|^{\frac{1}{2}} \times cc = \overline{\frac{b}{c}}\Big|^{\frac{1}{2}} \times \overline{c^4}\big|^{\frac{1}{2}} = \overline{bc^3}\big|^{\frac{1}{2}}$ is equal the Product of the Extreams.

And

And $\overline{\frac{c}{b}}\Big|^{\frac{3}{2}} \times bb = \overline{\frac{ccc}{bbb}}\Big|^{\frac{1}{2}} \times \overline{b^4}|^{\frac{1}{2}} = \overline{c^3 b}|^{\frac{1}{2}}$ is equal to the Product of the Means:

And the former Product is equal to the latter, which proves, in some Measure, our Rule.

3. To find whether $b^{\frac{1}{n}}$ and $\overline{c^n b}|^{\frac{1}{n}}$ be commensurable, as also their Mutual Proportion.

The greatest common Divisor of $b^{\frac{1}{n}}$ and $\overline{c^n b}|^{\frac{1}{n}}$ is $b^{\frac{1}{n}}$, by which both of them being Divided, the Quotients are $1^{\frac{1}{n}}$ and $\overline{c^n}|^{\frac{1}{n}}$, which are Rational Quantities, and equal to 1 and c respectively; consequently the propos'd Surds are Commensurable, and $b^{\frac{1}{n}} : \overline{c^n b}|^{\frac{1}{n}} :: 1 : c$.

CHAP. III.

Multiplication of Simple Surds.

Rule.

IF the Surds be not of the same kind; that is, if they have different Indices; reduce them to one and the least kind; *viz.* to their least common Index (by *Prop.* 1. of the last Chap.) then, if the Surds be not Imaginary, Multiply them by one another without their Indices; and lastly, annex the least common Index to the Product. So this new Root shall be the Product sought. But if the propos'd Surds be Imaginary; See the following Remark.

Examples

So $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. Also $b^{\frac{1}{4}} \times c^{\frac{1}{4}}$ is $= \overline{bc}|^{\frac{1}{4}}$. And Universally $a^{\frac{1}{n}} \times z^{\frac{1}{n}} = \overline{az}|^{\frac{1}{n}}$.

Again $2^{\frac{1}{2}} \times 3^{\frac{1}{3}}$ is (by *Prop.* 1. of the last Chap.) $= 8^{\frac{1}{6}} \times 9^{\frac{1}{6}} = 72^{\frac{1}{6}}$.

Also $a^{\frac{1}{3}} \times b^{\frac{1}{4}}$ is (by *Prop.* 1. of the last Chap.) $= \overline{a^4}|^{\frac{1}{12}} \times \overline{b^3}|^{\frac{1}{12}} = \overline{a^4 b^3}|^{\frac{1}{12}}$.

And

And Universally $x^{\frac{1}{n}} \times y^{\frac{1}{n}}$ is (by *Prop.* of the last Chap.) $=$ $\overline{x^n}|^{\frac{1}{mn}} \times \overline{y^m}|^{\frac{1}{mn}} = \overline{x^n y^m}|^{\frac{1}{nm}}$.

Scholia.

1. When a Surd is to be Multiplyed by a Rational Quantity, it will be sometimes convenient to connex them with the Sign $\times$; so the Product of a and $b^{\frac{1}{n}}$ may be writ thus $a \times b^{\frac{1}{n}}$ or thus $a\sqrt[n]{b}$.

2. When two Rational Quantities are join'd with the Sign $\times$ to two Surds of the same kind. Multiply the Rational Part by the Rational, and the Surd Part by the Surd; and the two Products join'd together with the Sign $\times$ is the Product required.

So $a \times \overline{b}|^{\frac{1}{n}}$ into $c \times d^{\frac{1}{n}}$ is $= ac \times \overline{bd}|^{\frac{1}{n}}$: For $a \times b^{\frac{1}{n}}$ is $= \overline{a^n b}|^{\frac{1}{n}}$, and $c \times d^{\frac{1}{n}}$ is $= \overline{c^n d}|^{\frac{1}{n}}$; but $\overline{a^n b}|^{\frac{1}{n}} \times \overline{c^n d}|^{\frac{1}{n}} = \overline{a^n c^n bd}|^{\frac{1}{n}}$ which is manifestly $= ac \times \overline{bd}|^{\frac{1}{n}}$.

3. When any Surd is to be rais'd to any given Power; Multiply the Index of the Surd by the Quantity that denotes that Power:

Thus $a^{\frac{1}{n}}$ rais'd to the n^{th}-Power is $= a^{\frac{1}{n} \times n} = a^{\frac{n}{n}} = a$.

Also $a^{\frac{1}{n}}$ rais'd to the m^{th} Power is $= a^{\frac{1}{n} \times m} = a^{\frac{m}{n}}$.

* *Here Note that the Square of* $\pm\sqrt{-a}$ *is* $-a$; *and Universally, that* $\sqrt[n]{-a}$ *rais'd to the* n^{th}*-Power is* $= -a$.

Therefore $b\sqrt{-a} \times c\sqrt{-a}$ is $= bc \times -a = -bca$;
Also $b\sqrt{-a} \times -c\sqrt{-a}$ is $= -bc \times -a = +bca$;
Likewise $-b\sqrt{-a} \times -c\sqrt{-a} = +bc \times -a = -bca$;
Again $\sqrt{-ba} \times \sqrt{-ca} = \sqrt{b} \times \sqrt{-a} \times \sqrt{c} \times \sqrt{-a} = \sqrt{bc} \times -a = -a\sqrt{bc}$:
Also $\sqrt{-ba} \times -\sqrt{-ca} = \sqrt{b} \times \sqrt{-a} \times -\sqrt{c} \times \sqrt{-a} = -\sqrt{bc} \times -a = a\sqrt{bc}$:
Likewise $-\sqrt{-ba} \times -\sqrt{-ca} = -\sqrt{b} \times \sqrt{-a} \times -\sqrt{c} \times \sqrt{-a} = \sqrt{bc} \times -a = -a\sqrt{bc}$.
Again $b\sqrt[4]{-a} \times c\sqrt[4]{-a} = bc\sqrt{-a}$.

Note, also that $\sqrt[3]{-a}$, $\sqrt[5]{-a}$, $\sqrt[7]{-a}$, &c. *are equal to* $-\sqrt[3]{a}$, $-\sqrt[5]{a}$, $-\sqrt[7]{a}$, &c. *respectively: Consequently* $-\sqrt[3]{-a}$, $-\sqrt[5]{-a}$, $-\sqrt[7]{-a}$, &c. *are equal to* $\sqrt[3]{a}$, $\sqrt[5]{a}$, $\sqrt[7]{a}$, &c. *respectively.*

CHAP. IV.

Division of Simple Surds.

Rule.

IF the Surds be not of the ſame kind, firſt reduce them to the ſame kind (by *Prop.* 1. Chap. II.) Then, if neither of the Surds be Imaginary, Divide the Dividend by the Diviſor, without their Indices, and Superior to the Quotient place the Common Index; and this new Surd is the Quotient ſought: But if either of the Surds be Imaginary, ſee the Remarks in the laſt and in this Chap.

So $\sqrt{20}$ Divided by $\sqrt{5}$, gives $\sqrt{4}$ for a Quotient.

Alſo $\overline{ab}|^{\frac{1}{4}} \div \overline{ca}|^{\frac{1}{4}} = \overline{\frac{b}{c}}|^{\frac{1}{4}}$.

And Univerſally $z^{\frac{1}{p}} \div y^{\frac{1}{q}} =$ (by *Prop.* 1. Chap. II.) $\overline{z^q}|^{\frac{1}{pq}} \div \overline{y^p}|^{\frac{1}{pq}} = \overline{\frac{z^q}{y^p}}|^{\frac{1}{pq}}$.

When the Dividend and Diviſor are two Rational Quantities prefixt to two Surds of the ſame kind; Divide the Rational Part of the Dividend by the Rational Part of the Diviſor, as alſo the Surd Part of the Dividend by the Surd Part of the Diviſor; and the two Quotients connexed together with (or ſometimes without) the Sign $\times$ is the Quotient deſired.

So

So $ab\sqrt[m]{cd} \div bd\sqrt[m]{c}$ is $= \frac{a}{d}\sqrt[m]{d}$.

Also $ab \times \overline{cd}|^{\frac{1}{m}} \div b \times \overline{cd}|^{\frac{1}{m}}$ is $= a \times \overline{1}|^{\frac{1}{m}} = a$.

** *Note, that* $-a \div \sqrt{-a}$ *is* $= \sqrt{-a}$: *Also* $-a \div -\sqrt{-a}$ *is* $= -\sqrt{-a}$:
Likewise $a \div \sqrt{-a}$ *is* $= -\sqrt{-a}$: *Also* $a \div -\sqrt{-a}$ *is* $= \sqrt{-a}$:
Again $bca \div b\sqrt{-a} = -c\sqrt{-a}$.

CHAP. V. and VI.

Addition and Subtraction of Simple Surds.

Rule.

Divide the propos'd Surds by their greatest common Divisor, and if the Quotients be Rational (that is to say, If the propos'd Surds be Commensurable) Multiply the Sum of the Quotients by the said greatest common Divisor, and the Product shall be the Sum of the Surds propos'd: Or Multiply the Difference of the Rational Quotients by the said greatest common Divisor, and the Product shall be the Difference of the two Surds propos'd.

Examples.

1. Let it be required to add $\sqrt{8}$ to $\sqrt{32}$.

Their greatest common Divisor is $\sqrt{8}$, by which each of the propos'd Surds being Divided, the Quotients are $\sqrt{1}$ and $\sqrt{4}$, which are Rational Numbers, and equal to 1 and 2; the Sum of which is 3, which being Multiplyed by the greatest common Divisor ($\sqrt{8}$) produces $3\sqrt{8}$ ($= \sqrt{9} \times \sqrt{8} = \sqrt{72}$) which is $= \sqrt{8} + \sqrt{32}$.

2. I demand the Sum of $\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}|^{\frac{3}{2}}$?

$\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}|^{\frac{3}{2}}$ reduc'd to a common Index are equal to $\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\frac{c^3}{b^3}}|^{\frac{1}{2}}$: And, these reduc'd to a common Denominator, are equal

to $\overline{\frac{b^4}{cb^3}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c^4}{cb^3}}\Big|^{\frac{1}{2}}$, whose greatest common Divisor is $\overline{\frac{1}{cb^3}}\Big|^{\frac{1}{2}}$, by which each of them being Divided, the Quotients will be $\overline{b^4}|^{\frac{1}{2}}$ and $\overline{c^4}|^{\frac{1}{2}}$ which are Rational Quantities, and equal to b^2 and c^2: The Sum of which Quotients is b^2+c^2, which, being Multiplyed by the greatest common Divisor, will give $:b^2+c^2:\times\overline{\frac{1}{cb^3}}\Big|^{\frac{1}{2}}$, which is $=\overline{\frac{b}{c}}\Big|^{\frac{1}{2}}+\overline{\frac{c}{b}}\Big|^{\frac{1}{2}}$.

3. I demand the Sum of $\overline{bd^n}|^{\frac{1}{n}}$ and $\overline{bc^n}|^{\frac{1}{n}}$?

Their greatest common Divisor is $b^{\frac{1}{n}}$, by which each of them being Divided, the Quotients are $\overline{d^n}|^{\frac{1}{n}}$ and $\overline{c^n}|^{\frac{1}{n}}$, which are Rational Quantities, and equal to d and c; the Sum of which is $d+c$, which being Multiplyed by the greatest common Divisor $b^{\frac{1}{n}}$, the Product is $:d+c:\times b^{\frac{1}{n}}$, which is $=\overline{bd^n}|^{\frac{1}{n}}+\overline{bc^n}|^{\frac{1}{n}}$.

This last Example, being Universal, will serve to Demonstrate our Rule.

Suppose $\overline{b}|^{\frac{1}{n}}=a$; then $\overline{bd^n}|^{\frac{1}{n}}$ is $=a\times\overline{d^n}|^{\frac{1}{n}}=ad$;

And $\overline{bc^n}|^{\frac{1}{n}}$ is $=a\times\overline{c^n}|^{\frac{1}{n}}=ac$: But $ad+ac=\overline{d+c}\times a$ is $=$ Sum of the two propos'd Quantities; consequently (Substituting $b^{\frac{1}{n}}$ for a) $\overline{d+c}\times b^{\frac{1}{n}}$ is equal to the Sum of the two propos'd Quantities. *W. W. D.*

In like Manner, if it were required to Subtract $b^{\frac{1}{n}}$ from $\overline{bc^n}|^{\frac{1}{n}}$ the Remainder will be found (by our Rule) to be $\overline{c-1}\times b^{\frac{1}{n}}$.

Or, if it were required to find the Value of $b^{\frac{1}{n}}-\overline{bc^n}|^{\frac{1}{n}}$ in other Terms, it will (by our Rule) be found $=\overline{1-c}\times b^{\frac{1}{n}}$.

When the Surds are Incommensurable, neither their Sum nor Difference can be exprest by any single Root; but they are to be

be added by $+$, and Subtracted by $-$; whence arise Surds Binomial and Residual:

Thus the Sum of $\sqrt{6}$ and $\sqrt{7}$ is $\sqrt{6}+\sqrt{7}$; and the Difference of $6^{\frac{1}{2}}$ and $7^{\frac{1}{2}}$ is $7^{\frac{1}{2}}-6^{\frac{1}{2}}$; &c.

But Universally the Sum of $b^{\frac{1}{n}}$ and $c^{\frac{1}{n}}$ is manifestly $= \overline{\overline{b^{\frac{1}{n}}+c^{\frac{1}{n}}}\big|^{n}}\Big|^{\frac{1}{n}}$, or the Difference of $b^{\frac{1}{n}}$ and $c^{\frac{1}{n}}$ is $= \overline{\overline{b^{\frac{1}{n}} \backsim c^{\frac{1}{n}}}\big|^{n}}\Big|^{\frac{1}{n}}$;

Wherefore $b^{\frac{1}{2}}+c^{\frac{1}{2}}$ is $= \overline{\overline{b^{\frac{1}{2}}+c^{\frac{1}{2}}}\big|^{2}}\Big|^{\frac{1}{2}} = \overline{b+c+2\sqrt{bc}}\big|^{\frac{1}{2}}$: Also $\sqrt{b}-\sqrt{c}$ is $=\sqrt{:b+c-2\sqrt{bc}}$: And $b^{\frac{1}{3}}+c^{\frac{1}{3}}$ is $= \overline{b+c+3b^{\frac{2}{3}}c^{\frac{1}{3}}+3b^{\frac{1}{3}}c^{\frac{2}{3}}}\big|^{\frac{1}{3}}$.

The Operations of Compound and Universal Surds are so easily deduc'd from what hath been said of Simple ones, that I do not think it worth while to insert them. Add to this, that 'tis very difficult to print them, which was one Motive I had for not publishing them in the first Edition of this Book.

PART VII.

Concerning the Nature of Equations, and how to prepare them for a Solution.

AN Equation is the mutual comparing of two equal Quantities differently expreſs'd, which are call'd the Parts or Sides thereof, and is uſually denoted with the Sign $=$ between them; and the ſingle Quantities of both Parts are called the Terms of the Equation.

Thus $a = c^3 + eg + d$ is an Equation, in which the ſingle Quantity a, one Part thereof, is equal to the Compound Quantity $c^3 + eg + d$ the other Part, and the Terms of both Parts are the ſingle Quantities a, c^3, eg and d.

Equations are to be conſider'd chiefly after two ways; *viz.* either as the laſt Concluſions to which you come in the Reſolution of Queſtions; or, as means by the help whereof you are to obtain final or ſingle Equations. An Equation of the former Kind, is compos'd only of one unknown Quantity, Involv'd with known ones, if the Queſtion be determin'd, and propoſes ſomething certain to be found out. But thoſe of the latter Kind, Involve two or more unknown Quantities, which, for that Reaſon, muſt be compar'd among one another, and ſo connected, that out of all theſe may emerge a new Equation, in which there is only one unknown Quantity, which we ſeek, mix'd with known Quantities; which unknown Quantity, that it may be the more eaſily diſcover'd, that Equation muſt be transform'd, moſt commonly various Ways, until it becomes the moſt Simple that it can: And when that is done, the Equation is ſaid to be reduc'd, or fitted for an Anſwer.

A ſingle Equation is two-fold, *viz.* Pure or Simple, or Affected or Compound.

A Pure or Simple ſingle Equation is that wherein the Quantity ſought (as ſuppoſe a) is expreſs'd by one Power only, as by the 1*ſt*, 2*d*, or 3*d*, &c. Power: Thus $2a + c = d - f$, and $ca^3 = d + b$ are Simple Equations.

An

An Adfected, or Compound Equation is that in which there are two or more different Degrees, or Powers of the Quantity sought (a), as in this Equation, $a^3+2a^2-ca=cd$, there are three different Powers of a, *viz.* a^3, a^2 and a^1.

When any Question is propos'd to be resolv'd, it is requisite that the true Design and Meaning thereof be fully and clearly comprehended, so as you may be able to place down all the Quantities concern'd in their due order; *viz.* all the substituted Letters in such order as the Nature of the Question requires. The next Thing to be done, is to consider whether the Question be limitted or not, that is, whether it admits of a determinate Number of Answers, or not; and to discover that, observe the following Rules.

Rule 1.

When the Number of the Quantities sought, exceeds the Number of the given Equations; most Questions of this Kind are capable of innumerable Answers.

Example.

Suppose a Question were propos'd thus. There are three such Numbers, that if the first be added to the second, their Sum will be 22; and if the second be added to the third, their Sum will be 46. What are those Numbers?

Let the three Numbers sought be represented by three Letters, thus;

Call the first a, the second e, and the third y.

Then, $a+e=22$, and $e+y=46$, according to the State of the Question.

Here the Number of the Quantities sought is three, *viz.* a, e and y, and the Number of the given Equations is but two; therefore this Question is not limitted, but admits of innumerable Answers; because, for any one of those three Letters, you may take any Number at Pleasure that is less than 22, which, with a little Consideration, will be easy to conceive.

Rule 2.

When the Number of the given Equations (not depending upon one another) are just as many as the Number of the Quantities

tities ſought; then is the Queſtion truly limitted; *viz.* each Quantity ſought hath a determinate Number of Values.

As for Inſtance, Let the aforeſaid Queſtion (with one Condition added to it) be propos'd thus:

There are three Numbers (a, e and y, as before). If the firſt be added to the ſecond, their Sum will be 22; if the ſecond be added to the third, their Sum will be 46; and if the firſt be added to the third, their Sum will be 36. What are thoſe Numbers?

That is, $a+e=22$; $e+y=46$; and $a+y=36$.

Now the Queſtion is perfectly limitted, each ſingle Quantity having but one ſingle Value; to wit $a=6$, $e=16$, and $y=30$.

N.B. *If the Number of the given Equations exceeds the Number of the Quantities ſought, they not only limit the Queſtion, but oftentimes render it impoſſible, by being propos'd inconſiſtent one to another.*

After you deduce, from the ſeveral Equations concern'd in the Queſtion, a ſingle Equation, it often happens that the Powers of the unknown Quantity therein are ſo mix'd and intangled with known ones, that it requires ſome Trouble and Skill to bring them to one Side of the Equation, and thoſe that are known to the other Side (ſtill keeping them to a juſt Equality), which, by the Writers of *Algebra*, is call'd the Reduction of Equations. The Methods of doing which I ſhall comprize in the following *Sect.* referring you for more Examples to Part XI.

CHAP. I.

The Reduction of Single Equations.

Sect. 1. Of Reduction by Addition.

REduction by Addition is grounded upon *Axiom* 1. and is only the Tranſpoſing (*viz.* the Removing) of one or more Negative Quantities from either Side of the Equation to the other Side, with the Sign + before each of them. As in theſe

Examples.

Examples.

Suppose	1	$a - b - 4 = d$
$1 + \overline{4}$	2	$a - b = d + 4$
$2 + b$	3	$a = d + 4 + b$

Note, when any absolute Number is register'd in the Margin, you must draw a Line over it, to distinguish it from the other Numbers: As $\overline{4}$ in the second Step of this Example.

Let	1	$aa - 7bc = dd - 2ba$
$1 + 7bc + 2ba$	2	$aa + 2ba = dd + 7bc$

Sect. 2. Reduction by Subtraction.

Reduction by Subtraction is grounded upon *Axiom* 2. and is perform'd by Transposing (or Removing) one or more Affirmative Quantities from either Side of the Equation to the other Side, with the Sign $-$ before each of them. As in these

Examples.

Suppose	1	$aa + dc + b = dd + 2ba$
$1 - 2ba$	2	$aa - 2ba + dc + b = dd$
$2 - dc$	3	$aa - 2ba + b = dd - dc$
$1 - b$	4	$aa - 2ba = dd - dc - b$

Let	1	$a^3 = cc + 3baa + 2da$
$1 - 3baa - 2da$	2	$a^3 - 3baa - 2da = cc$

Sect. 3. Reduction by Multiplication.

When any of the Terms of an Equation containing one or more Powers of the unknown Quantity, is, or are Fractions;

Multiply the Equation by the Denominator of any of these Fractions; and with the Equation thus produc'd, if need be, proceed as above directed: And so on, 'till all the Terms, containing the Power, or Powers of the unknown Quantity, are reduc'd into Integers.

Or Multiply the first Equation, by the Product of all the Denominators therein, and you'll reduce it into an Equation, wherein all the Terms will be Integers.

Examples.

Suppose	1	$\frac{a}{5} = \frac{b}{a}$
1×5	2	$a = \frac{5b}{a}$
$2 \times a$	3	$aa = 5b$

Let	1	$\frac{a+b}{c} = \frac{dd}{a-b}$
$1 \times \overline{a-b}$	2	$\frac{aa-bb}{c} = dd$
$2 \times c$	3	$aa - bb = cdd$

Let	1	$\frac{aa}{b} + gc + f = \frac{dm}{a}$
$1 \times ba$	2	$aaa + gcba + fba = dmb.$

Sect. 4. Reduction by Division.

When you find out any Divisor of, or any Quantity that will exactly Divide the Equation;

Divide the Equation by that Divisor, and you'll reduce it into lower Terms.

Note, *a simple Divisor may be easily discover'd, only by the Inspection of the Equation: But, in order to discover a Compound Divisor, and then to apply it as above directed, you are to reduce all the Terms of the Equation to one Side, making them equal to* 0,

Examples.

Suppose	1	$7baaa + 14bcaa = 49bcda$
$1 \div 7ba$	2	$aa + 2ca = 7cd$

Let	1	$a^3 + 2caa - baa = 3bca - bbc$
$1 - 3bca + bbc$	2	$a^3 + 2caa - baa - 3bca + bbc = 0$
$2 \div : a - b :$	3	$aa + 2ca - bc = 0$

When the greatest Power of the unknown Quantity is Multiplyed by any that is known;

Let the whole Equation be Divided by that known Quantity, that so the said Power may be cleared. As in these

Suppose

Suppose	1	$7da + 6a = bc - 3cd$
$1 \div \overline{7d+6}$	2	$a = \dfrac{bc - 3cd}{7d + 6}$

Let	1	$bba^3 - 2bbaa = bdc + bcc$
$1 \div bb$	2	$a^3 - 2aa = \dfrac{dc + cc}{b}$

Sect. 5. Reduction by Involution.

When any of the Powers of the unknown Quantity is affected, in one or more Terms of the Equation, with one or more Surd Indices, or Radical Signs, these Surds being in their most Simple Terms;

Transpose, if need be, such Terms of the Equation as are most fit for your Purpose: Then Involve each Side of this Equation according to the Denominator (1 being the Numerator) of the least Index of the Surds. Repeat these Directions, if need be, and the Power, or Powers of the unknown Quantity will, in some Cases, become Rational: But for an universal Method you must have Recourse to the *Scholium* in the latter End of this Part.

Examples.

Suppose	1	$\sqrt{a} = \sqrt{:d + c:}$
$1 \odot 2$	2	$a = d + c$

Let	1	$\sqrt[3]{:a + c:} = \sqrt{ba}$
$1 \odot 3$	2	$a + c = \sqrt{bbbaaa}$
$2 \odot 2$	3	$aa + 2ac + cc = b^3a^3$

Let	1	$\sqrt[3]{:3cba - c^3:} - c = \sqrt{ba}$
$1 + c$	2	$\sqrt[3]{:3cba - c^3:} = c + \sqrt{ba}$
$2 \odot 3$	3	$3cba - c^3 = c^3 + 3cc\sqrt{ba} + 3cba + ab\sqrt{ab}$
3 Transp.	4	$-2c^3 = :3cc + ab: \times \sqrt{ab}$
$4 \odot 2$	5	$+4c^6 = 9bc^4a + 6bbccaa + b^3a^3$

Sect. 6. Reduction by Evolution.

When one Side of an Equation contains only known Quantities, and the other Side, containing some Power or Powers of the unknown Quantity, has a Rational Root;

Evolve each Side of the Equation, according to the Index denominating that Root; and you'll reduce the Power or Powers of the unknown Quantity to lower Dimensions.

Examples.

Suppose	1	$aa = 36$
$1\ \omega\ 2$	2	$a = \sqrt{36} = 6$

Let	1	$a^3 = 27$
$1\ \omega\ 3$	2	$a = 3$

Suppose	1	$a^3 = b^3 + 3bbc + 3bcc + c^3$
$1\ \omega\ 3$	2	$a = b + c$

Let	1	$aa = bb - dd$
$1\ \omega\ 2$	2	$a = \sqrt{:bb - dd:}$

Suppose	1	$aa - 2ab + bb = cc + 2df - fg$
$1\ \omega\ 2$	2	$a - b = \sqrt{:cc + 2df - fg:}$

Here follows one Example of clearing Equations, wherein all the foregoing Reductions are promiscuously us'd, as occasion requires.

Suppose	1	$\sqrt{\frac{aa+3bb}{4}} - \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}}.\ a = ?$
$1 \times \sqrt{4c}$	2	$\sqrt{:caa + 3bbc:} - \sqrt{:caa - 3bbc:} = \sqrt{4baa}$
$2 \odot 2$	3	$caa + 3bbc - 2\sqrt{:caa + 3bbc:} \times \sqrt{:caa - 3bbc:} + caa - 3bbc = 4baa$
That is	4	$2caa - 2\sqrt{:cca^4 - 9ccb^4:} = 4baa$
For		$caa + 3bbc + caa - 3bbc = 2caa$
And		$2\sqrt{:caa+3bbc:} \times \sqrt{:caa-3bbc:} = 2\sqrt{:cca^4 - 9ccb^4:}$
$4 \div 2$	5	$caa - \sqrt{:cca^4 - 9ccb^4:} = 2baa$
$5 + \sqrt{:}$	6	$caa = 2baa + \sqrt{:cca^4 - 9ccb^4:}$
$6 - 2baa$	7	$caa - 2baa = \sqrt{:cca^4 - 9ccb^4:}$
$7 \odot 2$	8	$cca^4 - 4bca^4 + 4bba^4 = cca^4 - 9ccb^4$
8 Transp.	9	$9ccb^4 = 4bca^4 - 4bba^4$
$9 \div b$	10	$9ccb^3 = 4ca^4 - 4ba^4$
$10 \div \overline{4c-4b}$	11	$\frac{9ccbbb}{4c-4b} = a^4$
$11\ \omega\ 4$	12	$\sqrt[4]{\frac{9ccbbb}{4c-4b}} = a$, as was required.

By

By the Help of these Reductions (properly applyed) the unknown Quantity (a) or its Powers are cleared and brought to one Side of an Equation; and if the unknown Quantity (a) is found = those that are known, then the Question is answer'd, as in the 1*st.* *Examples* of *Sect.* 1*st*, 4*th*, 5*th*, and 6*th*.

Or, if any single Power of the unknown Quantity (a) is found equal to those that are known, then the respective Root of the known Quantities is the Answer, as in the three last Examples.

But when different Powers of the unknown Quantity are contain'd in an Equation, as $aa+ba=dd$, or $a^3-da^2=cf$, *&c.* then it is an adfected Equation; the Method of resolving which, that is of finding the Values of a therein has been shewn in Part IV. Chap. 3. And other Methods for the same Purpose shall be shewn further on.

CHAP. II.

How to reduce Equations, containing two or more unknown Quantities into a single Equation.

IT hath been shewn in the preceding Chap. how to reduce single Equations: But, in the Solution of Questions, these are frequently to be first deduc'd from others that contain two or more unknown Quantities. It will therefore be proper likewise to shew how to Exterminate by two, three or four, *&c.* Equations (concern'd in any limitted Question, and not depending upon one another) one, two or three, *&c.* unknown Quantities. But since upon the preceding *Sect.* these in this Chap. depend, it is proper that the foregoing should (as they are) be first treated off.

Sect. 1. The Extermination of one unknown Quantity by two Equations.

Case 1.

When the Quantity to be Exterminated is only of one Dimension in one of the Equations, its Value is to be sought in that Equation; and then this Value and its Powers to be Substituted for that Quantity and its respective Powers in the other Equation. Thus,

If $1 . a + b = e$, and $2 . aa + ca = d - ce$, that a may be Exterminated.

By 1*st*. $a = e - b$; then, by this Step and 2*d*. $: e - b : \times : e - b : + c \times : e - b : = d - ce$; which Step, being reduc'd, by Chap. I. gives $ee - 2be + 2ce = d - bb + cb$.

Case 2.

When the Quantity to be taken away is of two, or more Dimensions in both Equations, the Value of its greatest Power must be sought in both; then, if those Powers are not the same, the Equation that contains the lesser Power must be Multiplyed by the Quantity to be taken away, or by its Square, or Cube, &c. that it may become of the same Power it has in the other Equation. Then the Values of those Powers are to be made equal, and there will come out a new Equation, where the greatest Power or Dimension of the Quantity to be taken away is diminish'd.

And by repeating this Operation, the Quantity will, at length, be taken away; thus,

If 1. $aee + be + c = 0$, and 2. $fee + ge + h = 0$; and e is to be taken away.

Then, by the 1*st*. $- ee = : be + c : \div a$, and, by the 2*d*. $- ee = : ge + h : \div f$. By the two last Equations $: be + c : \div a \; (= - ee) = : ge + h : \div f$. This Equation Multiplyed by af gives $fbe + fc = age + ah$. And, by Transposition, and Division $^{*}e = : ah - fc : \div : fb - ag :$

Again, Multiplying the last Equation by $- e$, we have $- ee = : fce - ahe : \div : fb - ag :$ From the 3*d*. and last Equations $: be + c : \div a = : fce - ahe : \div : fb - ag :$ And, by Multiplication, Transposition and Division, $e = : agc - bcf : \div : aah - afc - agb + fbb :$. By the last Equation, and that mark'd with * we have $\dfrac{agc - fbc}{aah - afc - agb + fbb} = \dfrac{ah - fc}{fb - ag}$ which is an Equation exclusive of the Quantity e, as was required. This Equation, when reduc'd, by the last Chap. gives $hhaa - 2cfha - bgha + cgga + bbfh - bcfg + ccff = 0$.

Note, *By the first or second Equations and that mark'd with *, you may more expeditiously Exterminate the Quantity* e, *by* Case 1.

Note also, the Quantity e *may be Exterminated out of the preceding Example, by finding the Value of* e *in either of the first two Equations, by* Part X. *and then proceeding by* Case 1. *This in some Cases, happens to be a more concise Method than, but not so universal as, the foregoing one.*

Sect. 2. The Extermination of two or more unknown Quantities, by three or more Equations.

* In *Sect.* 1. we have discours'd of taking away one unknown Quantity by two Equations: But, if there be two, three or four, &c. Quantities to be taken away, there must be three, four or five, &c. Equations. And then the Business may be done by Degrees: As for Instance,

If $ba = ye$, $a + y = e$, and $5a = y + 3e$, that a and e may be exterminated.

First take away one of the Quantities a or e, suppose a, by substituting for its Value $ye \div b$ (found by the 1*st.* Equation) in the 2*d.* and 3*d.* Equations, and then you'll have $\frac{ye}{b} + y = e$, and $\frac{5ye}{b} = y + 3e$. Now, by these two last Equations you may easily take away e, as is taught in *Sect.* 1.

Scholium.

Hitherto may be referred the Extermination of Surd Quantities out of Equations, by making them equal to any Letters.

As, if you have $\sqrt{ay} - \sqrt{:aa - ay:} = 2a + \sqrt[3]{ayy}$; by writing v for $\sqrt{ay}$, w for $\sqrt{:aa - ay:}$, and x for $\sqrt[3]{ayy}$, you'll have the Equations $v - w = 2a + x$, $vv = ay$, and $ww = aa - ay$, and $x^3 = ayy$; out of which taking away, by Degrees, v, w and x, there will result an Equation intirely free from Surdity.

PART VIII.

Of Proportional Quantities, Arithmetical, Geometrical, and Musical.

CHAP. I.

Of Arithmetical Proportion.

Definition.

WHen any Rank or Series of Numbers or Quantities do either Increase or Decrease by an equal Interval, or common Difference or Excess, they are said to be in Arithmetical Progression, or Proportion continued:

$$\text{As}\begin{Bmatrix} 2 \,.\, 4 \,.\, 6 \,.\, 8 \,.\, 10 \,.\, \&c. \\ 15 \,.\, 13 \,.\, 11 \,.\, 9 \,.\, 7 \,.\, \&c. \end{Bmatrix}$$

In the former there is a continual Increase; in the latter a continual Decrease by 2, which is the common Difference or Excess.

And universally putting a for the first Term, and e for the common Excess or Difference, the Terms will be

$a,\ a+e,\ a+2e,\ a+3e,$ &c. Increasing.
$a,\ a-e,\ a-2e,\ a-3e,$ &c. Decreasing.

But the most Simple and Natural Progression is that which begins with 0, as

$0,\quad e,\quad 2e,\quad 3e,\quad 4e,\quad 5e,$ &c.
$0,\ -e,\ -2e,\ -3e,\ -4e,\ -5e,$ &c.

When it begins with any other Term (as a in the former Progressions) it is really a Compound of two Progressions; one of Equals ($a, a, a, a,$ &c.) and the other of Proportionals ($0, e, 2e, 3e,$ &c.). But

But when the first Term exceeds, or is exceeded by, the second Term by the same Number or Quantity, that the third exceeds, or is exceeded by, the fourth Term, but not by the same that the second exceeds, or is exceeded by the third Term; then that is said to be a Discontinued, or Disjunct Arithmetical Progression:

So $\left\{ \begin{matrix} 1, & 3; & 9, & 11 \\ a, & a+e, & a+4e, & a+5e \end{matrix} \right\}$ are said to be Disjunct or Discontinued Arithmetical Proportions.

Now in order to the finding out how to resolve Questions concerning these Progressions.

Let $\left\{ \begin{array}{l} a = \text{the least Term.} \\ y = \text{the greatest Term.} \\ e = \text{the common Excess.} \\ n = \text{the Number of Terms.} \\ s = \text{the Sum of all the Series, } viz. \text{ of all the Terms.} \end{array} \right.$

Lemma.

The Sum of the Extreams (*i. e.* of the least and greatest Terms) is equal to the Sum of any two Means that are equally distant from their Extreams (*i. e.* equal to the Sum of the least but one, and greatest but one, or least but two, and greatest but two, or least but three, and greatest but three, &c. Terms) of any Arithmetical Progression; and consequently, if the Number of Terms be odd, the double of the middle Term is equal to the Sum of the Extreams; or, if the Number of Terms be even, the Sum of the two middle Terms is equal to the Sum of the Extreams.

Demonstration.

$$\left.\begin{array}{r} \text{Least Term } a \\ + \text{ Greatest Term } y \end{array}\right\} = a + y$$
$$\left.\begin{array}{r} \text{Least Term but one } a + e \\ + \text{ Greatest Term but one } y - e \end{array}\right\} = a + y$$
$$\left.\begin{array}{r} \text{Least Term but two } a + 2e \\ + \text{ Greatest Term but two } y - 2e \end{array}\right\} = a + y$$
$$\left.\begin{array}{r} \text{Least Term but three } a + 3e \\ + \text{ Greatest Term but three } y - 3e \end{array}\right\} = a + y$$
$$\&c. \qquad = \&c. \quad Q. E. D.$$

N.B. *What follows relates to* $\div$.

Scholium

Scholium 1.

Whence it follows (and is very easy to conceive) that, if the Sum of the two Extreams be Multiplyed into the Number of all the Terms in the Series, that Product will be double the Sum of all the Series; that is to say $:a+y:\times n = 2s$.

Scholium 2.

In the foregoing $\left\{\begin{matrix}\text{Increasing}\\ \text{Decreasing}\end{matrix}\right\}$ Series, *viz.* $\left\{\begin{matrix} a,\ a+e,\ a+2e,\ a+3e,\ \&c.\\ y,\ y-e,\ y-2e,\ y-3e,\ \&c.\end{matrix}\right\}$ it is easy to perceive that the common Difference e is so often $\left\{\begin{matrix}\text{Added}\\ \text{Subtracted}\end{matrix}\right\}$ in the last Term of the Series, as are the Number of Terms except the first; that is to say, the first Term $\left\{\begin{matrix}a\\ y\end{matrix}\right\}$ hath no Difference $\left\{\begin{matrix}\text{Added}\\ \text{Subtracted}\end{matrix}\right\}$ in it; but the last Term hath $:n-1:$ times e $\left\{\begin{matrix}\text{Added}\\ \text{Subtracted}\end{matrix}\right\}$ in it; consequently the Difference of the Extreams is $= e$ into the Number of all the Terms less Unity or 1; that is, $y - a = e \times :n-1: = ne - e$.

Now by the help of these two Scholia, if any three of the aforesaid five Terms (*viz.* a, y, e, n, s) be given; the other two may be easily found;

Thus	1	$na+ny=2s$ As before.
And	2	$y-a=ne-e$
$1\div\overline{2}$	3	$\dfrac{na+ny}{2}=s$
$1\div\overline{a+y}$	4	$n=\dfrac{2s}{a+y}$
$1-ny$	5	$na=2s-ny$
$5\div n$	6	$a=\dfrac{2s-ny}{n}=\dfrac{2s}{n}-y$
$1-na$	7	$ny=2s-na$
$7\div n$	8	$y=\dfrac{2s}{n}-a$
$2+a$	9	$y=a+ne-e$

$2 \div \overline{n-1}$	10	$\frac{y-a}{n-1}=e$.
$9 - ne + e$	11	$y-ne+e=a$.
$2+e$	12	$y-a+e=ne$.
$12 \div e$	13	$\frac{y-a}{e}+1=n$.
$8=9$	14	$\frac{2s}{n}-a\,(=y)=a+ne-e$.
$14+a$	15	$\frac{2s}{n}=2a+ne-e$.
$15 \times n$	16	$2s=2na+nne-ne$.
$16 \div 2$	17	$s=\frac{2na+nne-ne}{2}$.
$\overline{4=13}$	18	$\frac{2s}{a+y}\,(=n)=\frac{y-a}{e}+1$.
$18 \times \overline{ae+ye}$	19	$2se=yy-aa+ae+ye$
$19 \div 2e$	20	$s=\frac{yy-aa+ae+ye}{2e}$.
$16 \div e$	21	$\frac{2s}{e}=nn+\frac{2a-e}{e}n$.
* Comp. □	22	$\frac{2s}{e}+\frac{aa-ae+\frac{1}{4}ee}{ee}=nn+\frac{2a-e}{e}n+\frac{aa-ae+\frac{1}{4}ee}{ee}$.
22 ɯ . 2.	23	$\frac{\sqrt{:2se+aa-ae+\frac{1}{4}ee:}}{e}=n+\frac{a-\frac{1}{2}e}{e}$.
$23-\frac{a}{e}+\frac{1}{2}$	24	$\frac{\sqrt{:2se+aa-ae+\frac{1}{4}ee:}-a}{e}+\frac{1}{2}=n$.
$19+aa-ae$	25	$2se+aa-ae=yy+ey$.
$*25+\frac{1}{4}ee$	26	$2se+aa-ae+\frac{1}{4}ee=yy+ey+\frac{1}{4}ee$.
26 ɯ . 2	27	$\sqrt{:2se+aa-ae+\frac{1}{4}ee:}=y+\frac{1}{2}e$.
$27-\frac{1}{2}e$	28	$\sqrt{:2se+aa-ae+\frac{1}{4}ee:}-\frac{1}{2}e=y$.
$25-aa-ye$	29	$2se-ye-ae=yy-aa$.
$29 \div$	30	$e=\frac{yy-aa}{2s-y-a}$.
$16-2na$	31	$2s-2na=nne-ne$.
$31 \div \overline{nn-n}$	32	$\frac{2s-2na}{nn-n}=e$.
$6=11$	33	$\frac{2s}{n}-y\,(=a)=y-ne+e$.
$33+y$	34	$\frac{2s}{n}=2y-ne+e$.

* See Part X.

$34 \times \frac{n}{2}$	35	$s = ny + \frac{1}{2}ne - \frac{1}{2}nne.$
$16 - nne + ne$	36	$2s - nne + ne = 2na.$
$36 \div 2n$	37	$\frac{2s - nne + ne}{2n} = a = \frac{s}{n} - \frac{1}{2}ne + \frac{1}{2}e.$
$34 + ne - e$	38	$\frac{2s}{n} + ne - e = 2y.$
$38 \div 2$	39	$\frac{s}{n} + \frac{1}{2}ne - \frac{1}{2}e = y.$
$26 - 2se$	40	$aa - ae + \frac{1}{4}ee = yy + ey + \frac{1}{4}ee - 2se.$
$40\ \omega\ .\ 2\ .$	41	$a - \frac{1}{2}e = \sqrt{:}\,yy + ey + \frac{1}{4}ee - 2se:$
$41 + \frac{1}{2}e$	42	$a = \frac{1}{2}e \pm \sqrt{:}\,yy + ey + \frac{1}{4}ee - 2se:$
$35 \times \frac{2}{e}$	43	$\frac{2s}{e} = \frac{2y + e}{e}n - nn.$
43 Tranſp.	44	$nn - \frac{2y + e}{e}n = -\frac{2s}{e}.$
* Comp. □	45	$nn - \frac{2y + e}{e}n + \frac{yy + ey + \frac{1}{4}ee}{ee} = \frac{yy + ey + \frac{1}{4}ee - 2se}{ee}.$
$45\ \omega\ .\ 2\ .$	46	$n - \frac{y + \frac{1}{2}e}{e} = \frac{\sqrt{:}\,yy + ey + \frac{1}{4}ee - 2se:}{e}$
$46 + \frac{y}{e} + \frac{1}{2}$	47	$n = \frac{y \pm \sqrt{:}\,yy + ey + \frac{1}{4}ee - 2se:}{e} + \frac{1}{2}.$
$35 - ny$	48	$s - ny = \frac{1}{2}ne - \frac{1}{2}nne.$
$48 \div \frac{n - nn}{2}$	49	$\frac{2s - 2ny}{n - nn} = e = \frac{2ny - 2s}{nn - n}.$

The Theorems by which any Queſtion in ÷ may be ſolv'd, are inſerted in the 3, 4, 6, 8, 9, 10, 11, 13, 17, 20, 24, 28, 30, 32, 35, 37, 39, 42, 47 and 49th Steps.

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* *See* Part X.

CHAP.

CHAP. II. Of Geometrical Proportions.

Sect. 1. Of Geometrical Proportions continued.

Definition.

WHen a Rank or Series of Numbers or Quantities, do either increase by one common Multiplicator, or decrease by one common Divisor, those Numbers or Quantities are said to be in Geometrical Proportion continued.

$$\text{As}\left\{\begin{array}{l} 2,\ 4,\ 8,\ 16,\ 32,\ \&c. \quad \text{Here 2 is the common Multiplyer.} \\ 729,\ 243,\ 81,\ 27,\ 9,\ \&c. \quad \text{Here 3 is the common Divisor.} \end{array}\right.$$

$$\text{Also}\left\{\begin{array}{l} 1,\ r,\ rr,\ r^3,\ r^4,\ \&c. \quad \text{Here } r \text{ is the common Multiplyer.} \\ 1,\ \frac{1}{r},\ \frac{1}{rr},\ \frac{1}{r^3},\ \frac{1}{r^4},\ \&c. \quad \text{Here } r \text{ is the common Divisor.} \end{array}\right.$$

$$\text{And Universally}\left\{\begin{array}{l} a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \&c. \quad \text{Here } r \text{ is the com. Mult.} \\ a,\ \frac{a}{r},\ \frac{a}{r^2},\ \frac{a}{r^3},\ \frac{a}{r^4},\ \&c. \quad \text{Here } r \text{ is the com. Div.} \end{array}\right.$$

Note, the common Multiplyer (or Divisor) is called the Ratio, and shews the Habitude or Relation the Numbers or Quantities have to one another; *viz.* whether they are Double, Treble, Quadruple, &c.

In order to solve such Questions as relate to Geometrical Proportion continued, I will

$$\text{Suppose}\left\{\begin{array}{l} a = \text{the least Term.} \\ y = \text{the greatest Term.} \\ s = \text{Sum of all the Terms.} \\ n = \text{the Number of Terms.} \\ r = \text{the common Ratio; but } \textit{Note} \text{ that } r \text{ must be } \sqsubset 1. \end{array}\right.$$

Lemma.

If from the Sum of any Series in $\div\div$ the least and greatest Terms be successively Subtracted; I say the first Remainder is equal to the Product of the second Remainder and common Ratio.

Demonstration.

a being $=$ the least Term of any Series in $\div\!\div$, and $r =$ the common Ratio; the Least, Least but one, Least but two, Least but three, &c. Terms will be equal to a, ar, ar^2, ar^3, &c. respectively:

Also y being $=$ the greatest Term of the said Series; the Greatest, Greatest but one, Greatest but two, Greatest but three, &c. Terms will be equal to y, $\frac{y}{r}$, $\frac{y}{r^2}$, $\frac{y}{r^3}$, &c. respectively:

And the Means, or middle Terms we will denote by &c. by which Means our Series in $\div\!\div \left\{\begin{matrix}\text{Increasing}\\ \text{Decreasing}\end{matrix}\right\}$ may be writ thus

$$\left\{\begin{matrix} a,\ ar,\ ar^2,\ ar^3,\ \&c.\ \frac{y}{r^3},\ \frac{y}{r^2},\ \frac{y}{r},\ y \\ y,\ \frac{y}{r},\ \frac{y}{r^2},\ \frac{y}{r^3},\ \&c.\ ar^3,\ ar^2,\ ar,\ a. \end{matrix}\right\}$$

Then $a + ar + ar^2 + ar^3 + \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y\,(=$

$$y + \frac{y}{r} + \frac{y}{r^2} + \frac{y}{r^3} + \&c. + ar^3 + ar^2 + ar + a) = s.$$

Wherefore, by Transposition,

$$ar + ar^2 + ar^3 + \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y = s - a$$

And $a + ar + ar^2 + ar^3 + \&c + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} = s - y$

But the first Part of the last Equation $\times r$ is manifestly equal to the first Part of the last but one Equation; Consequently $:s - y: \times r = s - a$.

Scholium.

It is manifest, by viewing the foregoing Series in $\div\!\div$ that ar^{n-1} is $= y$; for the Exponent of r in the least Term of that Series is 0, in the least Term but one is 1, in the least Term but two is 2, &c. and therefore universally the Exponent of r in the n^{th} or greatest Term of that Series is $n - 1$; wherefore $ar^{n-1} = y$.

By

By the help of this *Lemma* and *Scholium*, if any three of the five Terms, *viz.* a, y, s, n, r, be given, the other two may be found;

Thus	1	$rs - ry = s - a$ } As before.
And	2	$ar^{n-1} = y$
$1 \div \overline{s - y}$	3	$r = \frac{s - a}{s - y}$.
$1 + \overline{ry - s}$	4	$rs - s = ry - a$.
$4 \div \overline{r - 1}$	5	$s = \frac{ry - a}{r - 1} = y + \frac{y - a}{r - 1}$.
$4 + a$	6	$rs - s + a = ry$.
$6 \div r$	7	$\frac{rs + a - s}{r} = y = s + \frac{a - s}{r}$.
$6 - rs + s$	8	$a = ry - rs + s$.
$2 \div r^{n-1}$	9	$a = \frac{y}{r^{n-1}}$.
$2 \div a$	10	$r^{n-1} = \frac{y}{a}$.
$10\ w.\ \overline{n - 1}$	11	$r = \left(\frac{y}{a}\right)^{\frac{1}{n-1}}$
$10 \times r$	12	$r^n = \frac{yr}{a}$.
By the Log.	13	$n \times Lr = Ly + Lr - La$. { *Note*, L *stands for* Logarithm.
$13 \div Lr$	14	$n = \frac{Ly + Lr - La}{Lr} = \frac{Ly - La}{Lr} + 1$.
5, 2	15	$s = \frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1}$.
$2 = 7$	16	$ar^{n-1}\ (= y) = \frac{rs - s + a}{r}$.
$16 \times \frac{r}{a}$	17	$r^n = \frac{rs - s + a}{a} = \frac{rs - s}{a} + 1$.
By the Log.	18	$n \times Lr = L{:}\,rs - s + a{:} - La$.
$18 \div Lr$	19	$n = \frac{L{:}\,rs - s + a{:} - La}{Lr}$.
14, 3, and the L	20	$n = \frac{Ly - La}{L{:}\,s - a{:} - L{:}\,s - y{:}} + 1$.

$17 - \frac{rs}{a}$	21	$r^n - \frac{rs}{a} = 1 - \frac{s}{a}$.
2, 3	22	$a \times \overline{\frac{s-a}{s-y}}\Big\vert^{n-1} = y$.
$22 \times$	23	$a \times \overline{s-a}\vert^{n-1} = y \times \overline{s-y}\vert^{n-1}$.
5, 11	24	$s = y + \frac{y-a}{\overline{\frac{y}{a}}\Big\vert^{\frac{1}{n-1}} - 1}$
5, 9	25	$s = \frac{yr - \frac{y}{r^{n-1}}}{r-1} = \frac{yr^n - y}{r^n - r^{n-1}}$
$17 - 1$	26	$r^n - 1 = \frac{rs - s}{a}$.
$26 \times \frac{a}{r^n - 1}$	27	$a = \frac{rs - s}{r^n - 1}$.
$25 \times$	28	$sr^n - sr^{n-1} = yr^n - y$.
$28 \div$	29	$\frac{sr^n - sr^{n-1}}{r^n - 1} = y$.
$y \div 8 = 10$	30	$\frac{y}{ry - rs + s} \left(= \frac{y}{a}\right) = r^{n-1}$.
By the Log.	31	$Ly - L:ry - rs + s: = :n-1: \times Lr = nLr - Lr$.
$31 + Lr$	32	$Ly - L:ry - rs + s: + Lr = nLr$.
$32 \div Lr$	33	$\frac{Ly - L:ry - rs + s:}{Lr} + 1 = n$.
$28 - yr^n$	34	$sr^n - yr^n - sr^{n-1} = -y$.
$34 \div :s-y:$	35	$r^n - \frac{s}{s-y} r^{n-1} = - \frac{y}{s-y}$: An Equation

The Values of r and y sought in the Equations in the 21st, and 23d Steps are to be found by the converging Series.

wherein r is = the Root sought, and in which n, s and y are the three Quantities suppos'd to be given.

Scholium

Scholium.

If a be $=$ a finite Number, and $n =$ an infinite (or infinitely great) Number; then (r being, by a finite Number, $\sqsubset 1$) y (By 2*d* Step) $= ar^{n-1}$ must be $=$ an infinite Number; and consequently $s = \frac{ar^n - a}{r - 1}$ (by 15th. Step) must be infinite. But, if y be $=$ a finite Number, and $n =$ an infinite one; then (r being $\sqsubset 1$) a (by 9*th.* Step) $= \frac{y}{r^{n-1}}$ will be $=$ an infinitely small Number, and the only way we have of Writing such a Number is by 0;

Wherefore $s = \frac{ry - a}{r - 1}$ (by the 5*th* Step) will, in this Case, be $= \frac{ry}{r - 1}$; that is $=$ a finite Number.

By this Theorem, Questions that are usually propos'd in infinite decreasing Geometrical Proportions are easily solv'd: As for Instance,

If it were required to find the Sum of this decreasing Geometrical Proportion $\frac{B}{b}$, $\frac{B}{b^2}$, $\frac{B}{b^3}$, &c. *in infinitum.*

Here $\frac{B}{b} = y$ and $b = r$; Whence $\frac{ry}{r - 1} = s = \frac{B}{b - 1}$.

Again a, y, s being given, in order to find r in an infinite decreasing Series in ∺

I find, in the 3*d.* Step, that, in a finite Series in ∺, $r = \frac{s - a}{s - y}$; Wherefore, in an infinite decreasing one $r = \frac{s}{s - y}$.

Sect. 2. Of Geometrical Proportion Disjunct.

When the first Term has the same Ratio to the second, that the third hath to the fourth Term; but not the same Ratio which the second hath to the third Term; that Proportion is said to be Disjunct or Discontinued.

So $\left\{\begin{matrix} 2, & 4, & 18, & 36 \\ 36, & 18, & 4, & 2 \end{matrix}\right\}$, or Universally a, ae, d, de are said to be in Geometrical Proportion Disjunct; for the Ratio which 2 hath to 4 (which is 2) 18 hath to 36; but 4 has not the same

same Ratio to 18: And the same Ratio which a hath to ae (which is e) d hath to de, but ae hath not the same Ratio to d.

Theorem.

In any Geometrical Proportion Disjunct, the Product of the Means is equal to the Product of the Extreams; that is to say, since a, ae, d, de may represent any four Quantities in $::$ *viz.* $a \cdot\cdot ae :: d \cdot\cdot de$; it is plain that $ae \times d$ is $= a \times de = ade$.

If four Quantities are proportional, they will also be proportional in Alternation, Inversion, Composition, Division, Conversion and Mixtly. *Eucl.* 5. *Def.* 12, 13, 14, 15, 16.

that is, If	1	$a \cdot\cdot b :: c \cdot\cdot d$ be in direct Proportion.
Then	2	$a \cdot\cdot c :: b \cdot\cdot d$. Alternate. For $ad = bc$.
And	3	$b \cdot\cdot a :: d \cdot\cdot c$. Inverted. For $ad = bc$.
Also	4	$a + b \cdot\cdot b :: c + d \cdot\cdot d$. Compounded.
For	5	$da + db = bc + bd$; that is, $ad = bc$, as before.
Or	6	$a + c \cdot\cdot c :: b + d \cdot\cdot d$. Alternately Compounded.
For	7	$ad + cd = cb + cd$; that is $ad = cb$.
Again	8	$a - b \cdot\cdot b :: c - d \cdot\cdot d$. Divided.
For	9	$ad - bd = bc - bd$; that is $ad = cb$.
Or	10	$a - c \cdot\cdot c :: b - d \cdot\cdot d$. Alternately Divided.
For	11	$ad - cd = cb - cd$; that is $ad = cb$.
And	12	$a \cdot\cdot b \pm a :: c \cdot\cdot d \pm c$. Converted.
For	13	$ad \pm ac = bc \pm ac$; that is $ad = bc$.
Lastly	14	$a + b \cdot\cdot a - b :: c + d \cdot\cdot c - d$. Mixtly.
For	15	$ac - ad + bc - bd = ac + ad - bc - bd$.
that is	16	$2bc = 2ad$; Conseq. $bc = ad$; as at first.

Sect. 3. How to turn Equations into Analogies.

From the foregoing Section it will be easy to conceive how to turn or dissolve Equations into Anologies or Proportions:

For if the Rectangle of the two (or more) Quantities be equal to the Rectangle of two (or more) Quantities; then are those four (or more) Quantities proportional, by 16. 6. *Eucl. El.*

That is, if $ab = dc$; then is $a \cdot\cdot c :: d \cdot\cdot b$.
or $c \cdot\cdot a :: b \cdot\cdot d$, &c.

From

From whence there arises this general Rule for turning Equations into Analogies,

Rule.

Divide either Side of the given Equation (if it can be done) into two such Parts or Factors, as being Multiplyed together, will produce that Side again, and make those two Parts the two Extreams: Then Divide the other Side of the Equation (if it can be done) in the same Manner as the first was, and let those two Parts or Factors be the two Means.

For Instance, suppose $ab + ad = bd$.

Then $a \,..\, b :: d \,..\, b + d$, or $b \,..\, a :: b + d \,..\, d$, &c.

Or, taking ad from both Sides of the Equation, it will be, $ab = bd - ad$.

Then $a \,..\, d :: b - a \,..\, b$, or $b \,..\, d :: b - a \,..\, a$, &c.

Again, suppose $aa + 2ae = 2by + yy$.

Here a and $a + 2e$ are the two Factors of the first Side in this Equation; for $:a + 2e: \times a = aa + 2ae$. Again y and $2b + y$ are the two Factors of the other Side;

Therefore $a \,..\, y :: 2b + y \,..\, a + 2e$, or $2b + y \,..\, a + 2e :: a \,..\, y$, &c.

When one Side of any Equation cannot be Divided into two Factors as before, and the other Side can be so Divided; then make 1*st*. Unity and the former Side, or 2*dly*, the Square-Root of the said former Side, either the two Means or the two Extreams.

For Instance, Suppose $bc + bd = da + g$.

Then 1*st*. $b \,..\, 1 :: da + g \,..\, c + d$.

Or $1 \,..\, b :: c + d \,..\, da + g$, &c.

Or 2*dly*. $b \,..\, \sqrt{:da + g:} :: \sqrt{:da + g:} \,..\, c + d$.

Or $\sqrt{:da + g:} \,..\, b :: c + d \,..\, \sqrt{:da + g:}$, &c.

CHAP. III.

Of Harmonical Proportion.

Musical, or Harmonical Proportion, is when of three Quantities (or rather Numbers) the first hath the same Ratio to the third, as the Difference between the first and second, hath to the Difference between the second and third. As in these following.

Suppose a, b, c, in Musical Proportion.

Then	1	$a \,..\, c :: b - a \,..\, c - b$.
$\because$	2	$cb - ca = ac - ab$.
$2 + ca$	3	$cb = 2ac - ab$.
$3 \div \overline{2c - b}$	4	$\frac{cb}{2c - b} = a$ the first Term.
$3 + ba$	5	$cb + ab = 2ac$.
$5 \div \overline{c + a}$	6	$b = \frac{2ac}{c + a}$ the second Term.
$5 - cb$	7	$ab = 2ac - cb$.
$7 \div \overline{2a - b}$	8	$\frac{ab}{2a - b} = c$ the third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the fourth, as the Difference between the first and second, hath to the Difference between the third and fourth.

That is, Let a, b, c, d be the four Terms, &c.

Then	1	$a \,..\, d :: b - a \,..\, d - c$.
$\because$	2	$db - da = ad - ac$.
$2 + da$	3	$db = 2ad - ac$.
$3 \div \overline{2d - c}$	4	$\frac{db}{2d - c} = a$.
$3 \div d$	5	$b = 2a - \frac{ac}{d}$.
$3 + ac$	6	$db + ac = 2ad$.
$6 - db$	7	$ac = 2ad - db$.
$7 \div a$	8	$c = \frac{2ad - db}{a}$.
$7 \div \overline{2a - b}$	9	$\frac{ac}{2a - b} = d$.

PART

PART IX.

Of the Derivation and Composition of Equations.

THE first Sort of Equations which offer themselves to our Consideration are Laterals; such as is $a = b$, in which the Quantity a is determin'd to one single Value; *viz.* a in this Equation is equal to b, and to no other Quantity different from b in value.

But in the next Sort of Equations; *viz.* in Simple Quadraticks, as $a^2 = b$, it may be $a = +\sqrt{b}$, or $a = -\sqrt{b}$; for $-\sqrt{b} \times -\sqrt{b}$ is $= +b$ as well as $+\sqrt{b} \times +\sqrt{b}$.

Now if the known Quantities in the precedent Simple Quadratick, as also in the Values of a therein be transpos'd, you'll have $a^2 - b = 0$, $a - \sqrt{b} = 0$, and $a + \sqrt{b} = 0$; the first of which Equations is Manifestly the Product of the two last.

Hence you have the first Hint of the Origine of Equations, which Hint, being applyed to all Sorts of Quadratick and Superiour Equations, will be found to succeed, as will appear further on.

Again, If $a^3 = k$, or $a^3 - k = 0$, we may easily conceive a to be $= \sqrt[3]{k}$, *viz.* $a - \sqrt[3]{k} = 0$; But can't, at first, imagine or conceive that it can have any other Value in that Equation: For such other Value can't be Affirmative, neither can it be Negative:

Now this Obstacle will be remov'd, and an additional Mystery reveal'd, by Dividing $a^3 - k = 0$ by $a - \sqrt[3]{k}\ (= 0)$; for the Quotient thus had, is $a^2 + a\sqrt[3]{k} + \sqrt[3]{kk} = 0$, in which Equation the two Values of a will be found, by Part X. to be the two imaginary Quantities, *viz.* $:-\frac{1}{2} + \sqrt{-\frac{3}{4}} : \times \sqrt[3]{k}$, And $:-\frac{1}{2} - \sqrt{-\frac{3}{4}} : \times \sqrt[3]{k}$ (for the Cube of either of these Values of a is $= k$); consequently $a + :\frac{1}{2} - \sqrt{-\frac{3}{4}} : \times \sqrt[3]{k} = 0$, And $a + :\frac{1}{2} + \sqrt{-\frac{3}{4}} : \times \sqrt[3]{k} = 0$: And the Product of these two Equations Multiplyed by $: a - \sqrt[3]{k} : = 0$ produces $a^3 - k = 0$, as at first.

Furthermore any Equation wherein $+1$ is the Co-efficient of the highest Power of a the Root sought and o one of the Parts thereof may be represented by $a^n + ba^{n-1} + ca^{n-2}$, &c. $- h = 0$ (for b, c, &c. being Symbols or Universal, may therefore be equal to any Affirmative or Negative Numbers or Quantities); which Equation, being Multiplyed by $: a \mp k : = 0$, will produce the Equation $a^{n+1} + ba^n + ca^{n-1}$, &c. $- ha$ $\mp ka^n \mp bka^{n-1} \mp cka^{n-2}$, &c. $\pm hk = 0$. Now 'tis plain that if $\pm k$ be Substituted for and instead of a in the last Equation, the Result will be $\pm k^{n+1} \pm bk^n \pm ck^{n-1}$, &c. $\mp hk \mp k^{n+1} \mp bk^n \mp ck^{n-1}$, &c. $\pm hk = 0$, viz. $0 = 0$; wherefore one of the Values of a in the said last Equation is $\pm k$; consequently $a \mp k = 0$.

Hence we have strong Grounds to suppose. 1. That any Equation whatever, wherein $+1$ is the Co-efficient of the highest Power of a the Root sought, and o one of the Parts thereof, is produc'd by Subtracting the several Values of a therein from a (Where *Note*, that to Subtract $-$ is the same thing as to Add $+$) And Equating each Remainder to o, and then Multiplying all these Equations together: 2. Consequently that $a -$ any such Value of a $(= 0)$ will Divide any such Equation without leaving any Remainder: And, 3. That in such an Equation a hath so many Values, real or imaginary, as there are Units in the Index of its highest Power in that Equation.

But in order to put these three Suppositions beyond all doubt, &c. I will now consider them more particularly in like manner (I presume) as the Sagacious Author Mr. *Harriot* hath done.

CHAP. I.

The Origine of Quadratick Equations is thus deriv'd by Mr. *Harriot*.

Case 1. If $\left\{\begin{matrix} a = +b \\ a = -c \end{matrix}\right\}$; then by Transposition, $\left\{\begin{matrix} a - b = 0 \\ a + c = 0 \end{matrix}\right\}$, and, by Multiplying one by the other, you will have

$$aa + ca - ba - bc = 0;\ \text{that is}\ aa + : c - b : \times a - bc = 0.$$

Case

Case 2. If $\left\{\begin{matrix} a=-b \\ a=+c \end{matrix}\right\}$; then by Transposition, $\left\{\begin{matrix} a+b=0 \\ a-c=0 \end{matrix}\right\}$, and, by Multiplication,

$aa-ca+ba-bc=0$; that is $aa-:c-b:\times a-bc=0$.

Case 3. If $\left\{\begin{matrix} a=+b \\ a=+c \end{matrix}\right\}$; then by Transposition, $\left\{\begin{matrix} a-b=0 \\ a-c=0 \end{matrix}\right\}$, and, by Multiplication,

$$aa-:c+b:\times a+bc=0.$$

Case 4. If $\left\{\begin{matrix} a=-b \\ a=-c \end{matrix}\right\}$; then by Transposition, $\left\{\begin{matrix} a+b=0 \\ a+c=0 \end{matrix}\right\}$, and, by Multiplication,

$$aa+:c+b:\times a+bc=0.$$

Hence all Quadratick Equations, wherein the highest Term is a^2 or the Square of the Quantity sought, are reducible to these four Forms; *viz.*

1. $a^2+:c-b:\times a-bc=0.$
2. $aa-:c-b:\times a-bc=0.$
3. $aa-:c+b:\times a+bc=0.$
4. $aa+:c+b:\times a+bc=0.$

Note, That b *and* c, *being Symbols, may be either of them, equal to any Number or Quantity whatsoever, either Real or Imaginary, Simple or Compound.*

And these Mr. *Harriot* properly calls original Equations, and from them shews that every Quadratick Equation, wherein the highest Term is a^2, or the Square of the Quantity sought, hath just two Roots according to the Dimensions of the highest Power, as being made up by the Multiplication of two Lateral Equations: And these two Roots may be one of them Affirmative, and the other Negative, or both of them Affirmative, or both Negative, or both Imaginary: and sometimes they are equal to each other. (*Here Note*, when they are so with contrary Signs, the Equation will become a Simple Quadratick) and sometimes not: And the absolute Number $= bc$ is always = the Rectangle of the two Roots b and c (or of the two Values of a): And if it have a possitive Sign, the two Roots have like Signs; but if a Negative one, unlike: And the Co-efficient of a in the second Term is always equal to the Sum of both the Roots with contrary Signs.

☞ That b and c with their proper Signs, are the true Roots of the foregoing original Equations will appear by substituting b and c severally with their proper Signs, and their Squares, for and instead of a and its Square in each of them.

To instance what I here say, I will make use of the first original Equation, namely $aa + : c - b : \times a - bc = 0$; in which, if you substitute bb for aa, and b for a, it will become $bb + : c - b : \times b - bc = 0$, which is manifestly true; for by abbreviating this Equation all its Terms are destroyed, and it becomes $0 = 0$: Whence it is certain that one of the Roots of (or Values of a in) this Equation $aa + : c - b : \times a - bc = 0$ is b.

In the next place, I say $- c$ is the other Root of the said original Equation; for by substituting $+ cc$ (i. e. the Square of $- c$) for aa and $- c$ for a in that original Equation, it will become $cc + : c - b : \times - c - bc = 0$; that is, $cc - cc + bc - bc = 0$, or $0 = 0$.

But in the Equation $aa + : c - b : \times a - bc = 0$, a can't be equal to any other Quantity besides $+ b$ and $- c$: For supposing a equal to any Quantity either greater or less than b, as $b + z$ I say $b + z$ is $= - c$

Demonstration.

a being, by Supposition, $= b + z$, therefore $aa + : c - b : \times a - bc = 0$ will become $bb + 2bz + zz + cb + cz - bb - bz - bc = 0$; that is, $bz + zz + cz = 0$: wherefore $b + z + c = 0$, by Dividing each part by z (Viz. by $z \mp 0 = 0$); consequently $b + z = - c$. Q. E. D.

In like manner you may prove b and c with their proper Signs to be two Roots of any of the other three original Quadratick Equations.

Now supposing $c \sqsubset b$, and $c - b = p$, as also $c + b = q$, and $bc = h$, the four precedent Equations will become equal to these four following ones respectively.

$$1.\ aa + pa - h = 0. \qquad 3.\ aa - qa + h = 0.$$
$$2.\ aa - pa - h = 0. \qquad 4.\ aa + qa + h = 0.$$

In the first of which Cases you have the Sum of the two Roots sought $(s) = - p$; in the 2d. $s = + p$; in the 3d Case, $s = + q$, and in the 4th Case, $s = - q$: as also their Rectangle $(r) = - h$ in the 1st and 2d Cases, but $= + h$ in the 3d and 4th Cases, given in order to find their Difference (d); and then each of the said Roots.

The Method of finding all which shall be shewed in Part X.

Now,

Now, forasmuch as some Algebrists define Adfected Quadratick Equations in this manner; *Viz.*

" When the Quantity sought is brought to an Equality with " those that are known, and is at one side of the Equation in " no more than two different Powers whose Indices are double " to one another; those Equations are called Adfected Quadratick " Equations, "

We may say, by that Definition, that $a^4 \pm ba^2 = \pm fg$, Or $a^6 \pm da^3 = \pm c$, or universally $a^n \pm sa^{\frac{n}{2}} = \pm R$ is an Adfected Quadratick Equation.

But these Equations having as many Roots Real or Imaginary, as there are Units in the Index of the highest Power of the Root sought, and being not produced by the Multiplication of two Lateral Equations (as the foregoing original Quadraticks are) can't be said to be origiual Adfected Quadratick Equations; but (since the Method of solving these is of the same Nature with that of solving the foregoing ones) may be call'd adfected Quadratick Equations.

CHAP. II.

Of the Origine of Cubick &c. Equations.

MR. *Harriot* shews the Original of a Cubick Equation to be derived from three Lateral Equations, reduced first to the form of Residuals or Binomials, and then multipled together; or else from one Quadratick Multiplied by a Lateral. Whence he deduces, that all Cubick Equations have three Roots Real or Imaginary, or as many as are the Dimensions of its highest Power, and no more.

Thus to form a Cubick Equation, let the three Roots be $\left\{\begin{array}{l} a=b \\ a=c \\ a=d \end{array}\right\}$; then by Transposition $\left\{\begin{array}{l} a-b=0 \\ a-c=0 \\ a-d=0 \end{array}\right\}$, and these three Residuals multiplied together will produce this Equation

$$a^3 \begin{array}{l} -b \\ -c \\ -d \end{array} a^2 \begin{array}{l} +bc \\ +bd \\ +cd \end{array} a - bcd = 0$$

In

In like manner he shews the Derivation of Biquadratick Equations to be from four Lateral Equations reduced (as above) to the Form of Residuals or Binomials, and then multiplied together; or else from a Cubick into a Lateral; or from one Quadratick into another; or from a Quadratick multiplied by two Laterals continually. Wherefore he justly infers, that every Biquadratick will have four Roots real or imaginary agreeable to the Dimensions of its highest Power, and no more.

Thus if the former Cubick be multiplied by $:a+f:=0$, This Biquadratick Equation will be produced, *Viz.*

$$a^4 \begin{matrix} -b \\ -c \\ -d \\ +f \end{matrix}\, a^3 \begin{matrix} +bc \\ +bd \\ +cd \\ -fb \\ -fc \\ -fd \end{matrix}\, a^2 \begin{matrix} -bcd \\ +fbc \\ +fbd \\ +fcd \end{matrix}\, a - fbcd = 0.$$

From which Original of these Equations 'tis plain, that after you discover the value of any one Root, you may depress the Equation a Dimension lower, by dividing it by such Root reduced to the form of a Residual or Binomial, as above.

Thus, if you find that one Root, or one a of the foregoing Equation is $=-f$; then divide the Equation by $:a+f:(=0)$, and it will bring it down to a Cubick; and that Cubick being again divided by $:a-b:(=0)$, or $:a-c:(=0)$, or $:a-d:(=0)$ will be depress'd into an original Quadratick, &c. And this is sometimes of good use to dissolve Compound Equations into their Components, as shall be shewed further on.

From this Method of Composition of these Equations 'tis also apparent of what Members each of the Co-efficients are made up.

For,

I. The Co-efficient of the second Term is always the Aggregate or Sum of all the Roots with contrary Signs. Thus in the above Cubick Equation, the Co-efficient of the second Term is $-b-c-d$: And $-b-c-d+f$ is the Co-efficient of the second Term of the above Biquadratick Equation. But $b+c+d$ in the former; and $b+c+d-f$ in the latter Equation, is the Sum of all the Roots; whence what we have here said is manifest.

Wherefore it follows, that if all the Negative Roots, secluding their Signs, be equal to all the Affirmative ones; (tho' not each

each to each respectively) then will the second Term quite vanish out of the Equation, and be wanting, because the Affirmatives and Negatives do mutually destroy each other. And *Vice Versa*; whenever the second Term is wanting in any of these Equations, the Roots are thus equal, and have contrary Signs.

II. The Co-efficient of the Third Term is the Aggregate of all the Rectangles made by the Multiplication of every pair of the Roots (with their proper Signs) as often as they can be taken, which in a Cubick is three, in a Biquadratick is six, in a 5th Power is ten, &c. according to the order of Triangular Numbers. Thus, in the third Term of the Cubick Equation before-mentioned, $bc + bd + cd$ the Co-efficient is the Aggregate of the three Rectangles of the Roots b, c and d, taken by Pairs.

And here if all the Negative Rectangles, secluding their Signs, are equal to all the Affirmative ones, they will destroy one another, and so the third Term will vanish or be wanting.

III. The Co-efficient of the fourth Term is the Aggregate of all the Solids made by the continual Multiplication of all the Ternaries, or every three Roots, with contrary Signs, &c. And so on *ad Infinitum*.

IV. As in Quadraticks the absolute Number, or Quantity given is always the Rectangle of the two Roots or Values of a, so in Cubicks, 'tis always the Solid of all the three Roots, with their Signs changed, one into another; and in Biquadraticks, of all the four Roots; &c.

From this Method of Composition of those Equations, with due Consideration, it will be evident that,

1*st*. The Affirmative Roots of any Equation are changed into Negatives, and the Negative Roots into Affirmatives, by changing the Signs of every other Term of the Equation, that is the Signs of the 2*d*, 4*th*, 6*th*, 8*th*, &c. Terms, or of the 1*st*, 3*d*, 5*th*, 7*th*, &c. Thus the Signs of the Roots of this Equation $a^4 - a^3 - a^2 + 7a - 6 = 0$ (in which the Values of a are 1, -2, $1 + \sqrt{-2}$ and $1 - \sqrt{-2}$) are changed by writing it thus $a^4 + a^3 - a^2 - 7a - 6 = 0$, or thus $-a^4 - a^3 + a^2 + 7a + 6 = 0$ (in each of which two last Equations the Values of a are -1, $+2$, $-1 - \sqrt{-2}$, and $-1 + \sqrt{-2}$). And the Signs of the Roots of this Equation $a^3 + pa^2 + qa$

$+ r = 0$ are changed by writing it thus $a^3 - pa^2 + qa - r = 0$, or thus $- a^3 + pa^2 - qa + r = 0$.

And 2dly, The Number of imaginary Roots in any Equation whatever, not including any imaginary Term in it, is even; *viz.* none, two, four, six, or eight, &c.

PART

PART X.

The Solution of Adfected Quadratick Equations.

ALL Adfected Quadratick Equations do (as hath been already shewn) fall under the Consideration of these four Forms or Cases, *Viz.*

$$\left.\begin{array}{l} a^2+pa-b=0 \\ a^2-pa-b=0 \\ a^2-qa+b=0 \\ a^2+qa+b=0 \end{array}\right\} \text{or Universally} \left\{\begin{array}{l} a^n+pa^{\frac{n}{2}}-b=0 \\ a^n-pa^{\frac{n}{2}}-b=0 \\ a^n-qa^{\frac{n}{2}}+b=0 \\ a^n+qa^{\frac{n}{2}}+b=0 \end{array}\right.$$

When there happens to be more Terms in one of these kind of Equations than two, and the highest Power of the unknown Quantity is multiplied into some known Co-efficient, you must reduce them by Division (as in Part VII. Chap. 1. Sect. 4.) and for the fractional Quautities that may arise by those Divisions substitute another whole Quantity.

For Instance, let $baa+caa-da-ca=dc+cb$; then, by dividing by $b+c$, you'll have $aa-\frac{d+c}{b+c}a=\frac{dc+cb}{b+c}$

Now make $\frac{d+c}{b+c}=k$, and for $\frac{dc+cb}{b+c}$ put l; then will aa

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N. B. *The Values of* $+p$, $+q$ *and* $+b$ *are supposed each of them to be* = *Affirmative Numbers.*

$-ka=l$ be the new Equation equal to the other, and now fitted for a Solution. But here we will leave it, and return to our four preceeding Cases; in order to solve each of which, it will be requisite to premise this

Lemma.

Half (d) the Difference of any two Quantities (z and y, whereof z is the greater) added to half (s) their Sum, is equal to the greater; but subtracted from half their Sum, is equal to the Lesser of them.

Demonstration

	1	$\frac{1}{2}z+\frac{1}{2}y=\frac{1}{2}s$
	2	$\frac{1}{2}z-\frac{1}{2}y=\frac{1}{2}d$
$1+2$	3	$z=\frac{1}{2}s+\frac{1}{2}d$
$1-2$	4	$y=\frac{1}{2}s-\frac{1}{2}d$. Q.E.D.

It hath been already * proved, that in each of the four foregoing Forms or Cases of Adfected Quadratick Equations, the Sum (s) and Rectangle (r) of the two sought Roots are given, to find their Difference, and then the Roots themselves severally.

* *See Pag.* 110.

Now, by the foregoing *Lemma*,

$\frac{1}{2}s+\frac{1}{2}d=$ the greater of the sought Roots,

and $\frac{1}{2}s-\frac{1}{2}d=$ the lesser of them;

And, by multiplying them by one another, you'll have $\frac{1}{4}ss-\frac{1}{4}dd=r$

Then, by multiplying the last Equation by 4, you'll have $ss-dd=4r$

That is, by Transposition, $ss-4r=dd$

And by Evolution $\sqrt{:ss-4r:}=d$

Now, by our *Lemma*, $\frac{1}{2}s+\frac{\sqrt{:ss-4r:}}{2}=$ the greater of the sought Roots. And $\frac{1}{2}s-\frac{\sqrt{:ss-4r:}}{2}=$ the lesser of them:

Wherefore the Canon for finding the two Values of a in this 1st Case, *Viz.* $aa+pa-h=0$ is (because,* as we have already prov'd, $s=-p$, and $r=-h$ in this first Case) $-\frac{1}{2}p$ $\pm\frac{\sqrt{pp+4h}}{2}=\left\{\begin{array}{l}\text{Greater}\\ \text{Lesser, or Negative}\end{array}\right\}$ Value of a.

Also

Also the Canon for finding the two Values of a in this 2d Case, *viz.*

$aa - pa - b = 0$ is (because s is $= p$, and $r = -b$ in this 2d Case) $\frac{1}{2}p \pm \frac{\sqrt{pp + 4b}}{2} = \left\{\begin{matrix}\text{Greater} \\ \text{Lesser or Negative}\end{matrix}\right\}$ Value of a.

Again, the Canon for finding the two Values of a in this 3d Form or Case, *viz.*

$aa - qa + b = 0$ is (because s is $= q$, and $r = b$ in this 3d Case) $\frac{1}{2}q \pm \frac{\sqrt{:qq - 4b:}}{2} = \left\{\begin{matrix}\text{Greater} \\ \text{Lesser}\end{matrix}\right\}$ Value of a; both of which Values are Affirmative if $qq \sqsubset 4b$; and, for that Reason, this Equation is called Ambiguous; but if $qq \sqsupset 4b$, then those Values are Imaginary: For, in this latter Case, the Square-Root of a Negative Quantity is to be Extracted in order to find the Value of a: But a Negative Quantity hath no Square-Root; for whether the Root be Affirmative or Negative, its Square will be Affirmative; wherefore such impossible Equations are said to have Imaginary Roots.

And the Canon for finding the Values of a in this 4th Case, *viz.* $aa + qa + b = 0$, is (because $s = -q$, and $r = b$ in this 4th Case) $-\frac{1}{2}q \pm \frac{\sqrt{qq - 4b}}{2} = \left\{\begin{matrix}\text{Greater} \\ \text{Lesser}\end{matrix}\right\}$ Value of a, both of which Values are either Negative or Imaginary; and therefore this last Case is not taken of notice by many Algebrists; and I (for Brevities sake) will insert it no more.

Tho' the foregoing Method of Resolving Adfected Quadratick Equations is what naturally follows from Mr. *Harriot*'s Method of Compounding them, yet it is not the same with his, which is a peculiar Way of his own; and that is, after transposing b, by adding the Square of half the Co-efficient to each Part of the Equation, he thereby makes the unknown Part a compleat Square in Species.

Thus by Transposing the known or given Quantity b in the three first foregoing Forms or Cases, they will become

$$\left.\begin{matrix} 1.\ aa + pa = b \\ 2.\ aa - pa = b \\ 3.\ aa - qa = -b \end{matrix}\right\}\ \text{Respectively.}$$

And these Mr. *Harriot* calls Canonical Equations.

Now

Now 'tis manifest that if you add the Square of half the Co-efficient to both Sides of each Equation, the first Part of it will thereby become a perfect Square, by which Means the Values of the sought Root a will be made known; thus,

1. In the 1st Form where $aa + pa = b$.

Add the Square of $\frac{1}{2}p$, to wit $\frac{1}{4}pp$, to each Part, and the Sums are $aa + pa + \frac{1}{4}pp = b + \frac{1}{4}pp$.

Then by Extracting the Square-Root of each Part,

$a + \frac{1}{2}p = \pm \sqrt{:} b + \frac{1}{4}pp:$

And, by Transposing $\frac{1}{2}p$, you have

$a = -\frac{1}{2}p \pm \sqrt{:} b + \frac{1}{4}pp:$, as before.

2. In the 2d Form where $aa - pa = b$.

By compleating □, or adding the Square of $-\frac{1}{2}p$ to each Part,

$aa - pa + \frac{1}{4}pp = b + \frac{1}{4}pp$.

Then, by Extracting the Square-Root of each Part,

$a - \frac{1}{2}p = \pm \sqrt{:} b + \frac{1}{4}pp:$

And, by Transposition, $a = \frac{1}{2}p \pm \sqrt{:} b + \frac{1}{4}pp:$, as before.

3. In the 3d Case or Form, where $aa - qa = -b$,

Compleat the Square, by adding the Square of half the Co-efficient $-q$, that is $\frac{1}{4}qq$, to each Part, and you'll have

$aa - qa + \frac{1}{4}qq = -b + \frac{1}{4}qq:$

Then, by Extracting the Square-Root of each Part,

$a - \frac{1}{2}q = \pm \sqrt{:} -b + \frac{1}{4}qq:$

And, by Transposition, $a = \frac{1}{2}q + \sqrt{:} -b + \frac{1}{4}qq:$, as before.

Now by the Help of these Theorems it will be easy to calculate, or find the Values of the unknown Quantity (a) in Numbers.

Example 1. *Case* 1.

Suppose $aa + a = 6$;

Then $p = 1$ and $b = 6$; wherefore $a = -\frac{1}{2} \pm \sqrt{6\frac{1}{4}}$, by Theorem 1; that is $a = -\frac{1}{2} \pm 2\frac{1}{2} = \left\{ \begin{matrix} 2 \\ -3 \end{matrix} \right\}$; that is to say the two Values of a are $+2$ and -3. And either of these Roots being known, the other may be found by Subtracting the known one from the Co-efficient of the 2d Term with a contrary Sign, as is evident from the Composition of Affected Quadraticks.

So

So, if I know that $+2$ is one of the Roots of the above Equation $a^2+a=6$, I can have the other -3 by Subtracting $+2$ from the Co-efficient of the second Term with a contrary Sign, that is from -1: Or, if I had known that -3 is one of the Roots (or Values of a) in the said Equation, I can find the other $+2$ by Subtracting -3 from -1. Or if one of the Roots be known, the other may be had by Reducing all the Terms in the propos'd Equation to one Side, and Equating them to 0; and then Dividing them by $a -$ known Root $(=0)$, the Quotient will shew the other Root or Value of a; thus,

$$\begin{array}{l} a-2\ )\ aa+a-6=0\ \ (\ a+3=0 \\ \qquad\quad \underline{aa-2a} \\ \qquad\qquad\quad 3a-6 \\ \qquad\qquad\quad \underline{3a-6} \\ \qquad\qquad\qquad 0 \end{array}$$

$$\begin{array}{l} \text{Or } a+3\ )\ aa+a-6=0\ \ (\ a-2=0 \\ \qquad\qquad \underline{aa+3a} \\ \qquad\qquad\quad -2a-6 \\ \qquad\qquad\quad \underline{-2a-6} \\ \qquad\qquad\qquad 0 \end{array}$$

☞ *The Truth of this is manifest from the Composition of Original Adfected Quadraticks.*

And tho' it may seem impossible that the Value of a should be -3; *viz.* that an Affirmative Quantity shou'd be equal to a Negative One; yet by that Value and the same Co-efficient the true (or first) Eqnation may be form'd.

Thus, Let	1	$a=-3$
1 ⊗ 2	2	$aa=\quad 9$, *viz.* $=-3\times-3$
2 + 1	3	$aa+a=9-3=6$, as at first.

Ex-

Example 2. *Case* 2.

Suppose $aa - 7a = 948.75$. Then $-p = -7$, and $b = 948.75$. And, by *Canon*, or *Theorem* 2, $a = \frac{7}{2} \pm \sqrt{:948.75 + \frac{49}{4}:} = \frac{7}{2} \pm \sqrt{961} = 3\frac{1}{2} \pm 31 = \left\{\begin{matrix} 34.5 \\ -27.5 \end{matrix}\right\}$.

Now either of those Values of a being known, the other may be had by Subtraction or Division, as in *Exam.* 1.

Example 3. *Case* 3.

Suppose $aa - 36a = -243$. Here $-q = -36$, and $-b = -243$; then $a = 18 \pm \sqrt{:324 - 243:}$ by *Canon* 3; that is, $a = 18 \pm \sqrt{81} = \left\{\begin{matrix} 27 \\ 9 \end{matrix}\right\}$.

Now either of those Values of a being known, the other may be found by Subtraction or Division in like Manner as in *Exam.* 1.

Notwithstanding all Quadratick Equations of this third Form have two Affirmative (or Imaginary) Roots, yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question; for which Reason this third Form is call'd ambiguous.

From the foregoing Canons, by the Help of Substitution, others may be deduc'd, which will solve all Affected Quadratick Equations. Thus,

$$\text{If}\left|\begin{array}{c|l} 1 & a^n + pa^{\frac{n}{2}} = b \\ 2 & a^n - pa^{\frac{n}{2}} = b \\ 3 & a^n - qa^{\frac{n}{2}} = -b \end{array}\right.\left.\vphantom{\begin{array}{c}1\\2\\3\end{array}}\right\}\text{Then}$$

Suppose $a^{\frac{n}{2}} = e$, and the foregoing Equations will become

1

$$\left|\begin{array}{l|l} 1 & ee + pe = b \\ 2 & ee - pe = b \\ 3 & ee - qe = -b \end{array}\right\} \text{ Respectively; therefore}$$

$$\text{By } \textit{Canon} \left|\begin{array}{l|l} 1 & e = -\tfrac{1}{2}p \pm \surd : \tfrac{1}{4}pp + b : \\ 2 & e = \tfrac{1}{2}p \pm \surd : \tfrac{1}{4}pp + b : \\ 3 & e = \tfrac{1}{2}q \pm \surd : \tfrac{1}{4}qq - b : \end{array}\right\} \text{Respectively.}$$

But $e = a^{\frac{n}{2}}$ by Supposition; Consequently.

$$\left|\begin{array}{l|l} 1 & a = \sqrt[\frac{n}{2}]{} : -\tfrac{1}{2}p \pm \sqrt{\tfrac{1}{4}pp + b} : \\ 2 & a = \sqrt[\frac{n}{2}]{} : \tfrac{1}{2}p \pm \sqrt{\tfrac{1}{4}pp + b} : \\ 3 & a = \sqrt[\frac{n}{2}]{} : \tfrac{1}{2}q \pm \sqrt{\tfrac{1}{4}qq - b} : \end{array}\right\} \text{Respectively.}$$

There is another Method for Resolving Adfected Equations, and that is by casting of the second Term, which is done by Substitution, thus; Take always half the known Coefficient, and, in Cases 1st and 4th, add it to, but, in Cases 2d and 3d, Subtract it from its Fellow-factor, and for their Sum or Difference substitute another Letter. As in these

Case 1.

Let	1	$a^n + pa^{\frac{n}{2}} = b$
Put	2	$a^{\frac{n}{2}} + \frac{1}{2}p = e$
2 ⊙ 2	3	$a^n + pa^{\frac{n}{2}} + \frac{1}{4}pp = ee$
3 − 1	4	$\frac{1}{4}pp = ee - b$
4 + b	5	$\frac{1}{4}pp + b = ee$
5 ɯ 2	6	$\surd : \frac{1}{4}pp + b : = e$
2, 6	7	$a^{\frac{n}{2}} + \frac{1}{2}p \,(= e) = \surd : \frac{1}{4}pp + b :$
7 − $\frac{1}{2}p$	8	$a^{\frac{n}{2}} = -\frac{1}{2}p + \surd : \frac{1}{4}pp + b :$
8 ɯ $\frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{} : -\frac{1}{2}p + \sqrt{\frac{1}{4}pp + b} :$

Case 2.

Let	1	$a^{n} - pa^{\frac{n}{2}} = b.$
Put	2	$a^{\frac{n}{2}} - \frac{1}{2}p = e.$
$2 \odot 2$	3	$a^{n} - pa^{\frac{n}{2}} + \frac{1}{4}pp = ee.$
$3 - 1$	4	$\frac{1}{4}pp = ee - b.$
$4 + b$	5	$\frac{1}{4}pp + b = ee.$
$5 \, \omega \, 2$	6	$\sqrt{:\frac{1}{4}pp + b:} = e.$
$2, 6$	7	$a^{\frac{n}{2}} - \frac{1}{2}p \, (= e) = \sqrt{:\frac{1}{4}pp + b:}$
$7 + \frac{1}{2}p$	8	$a^{\frac{n}{2}} = \frac{1}{2}p + \sqrt{:\frac{1}{4}pp + b:}$
$8 \, \omega \frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{:\frac{1}{2}p + \sqrt{\frac{1}{4}pp + b}:}$

Case 3.

Let	1	$a^{n} - qa^{\frac{n}{2}} = -b.$
Put	2	$a^{\frac{n}{2}} - \frac{1}{2}q = e.$
$2 \odot 2$	3	$a^{n} - qa^{\frac{n}{2}} + \frac{1}{4}qq = ee.$
$3 - 1$	4	$\frac{1}{4}qq = ee + b.$
$4 - b$	5	$\frac{1}{4}qq - b = ee.$
$5 \, \omega \, 2$	6	$\pm\sqrt{:\frac{1}{4}qq - b:} = e.$
$2, 6$	7	$a^{\frac{n}{2}} - \frac{1}{2}q \, (= e) = \pm\sqrt{:\frac{1}{4}qq - b:}$
$7 + \frac{1}{2}q$	8	$a^{\frac{n}{2}} = \frac{1}{2}q \pm \sqrt{:\frac{1}{4}qq - b:}$
$8 \, \omega \frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{:\frac{1}{2}q \pm \sqrt{\frac{1}{4}qq - b}:}$

Hence 'tis evident that whatever Method is used in solving these (or indeed any other) Equations, the Result will still be the same, if the Work be true.

PART

PART XI.

Of Numerical Questions.

CHAP. I.

Questions producing Simple Equations.

Section 1. *Question* 1.

s (200) Pounds are to be Divided between two Men, that the one may have d (73) Pounds more than the other. What is the Share of each Man?

Suppose	1	$a =$ the greater Share.
And	2	$e =$ the Lesser.
Then	3	$a + e = s$ } by the Question.
and	4	$a - e = d$ }
$3 + 4$	5	$2a = s + d$.
$5 \div 2$	6	$a = \frac{s + d}{2}$ $(= \frac{200 + 73}{2}$ Pounds $= 136l. 10s. 00d.)$
$3 - 4$	7	$2e = s - d$.
$7 \div 2$	8	$e = \frac{s - d}{2}$ $(= 63l.\ 10s.\ 00d)$.

Or thus,

Suppose	1	$a =$ the greater Share.
Then	2	$a - d =$ the lesser Share } by the Question.
and	3	$a + a - d = 2a - d = s$ }
$3 + d$	4	$2a = s + d$.
$4 \div 2$	5	$a = \frac{s + d}{2} =$ greater Share.
2, 5	6	$\frac{s + d}{2} - d = \frac{s - d}{2} =$ lesser Share.

Question 2.

$$\left|\begin{array}{c|l} 1 & a + e = s\ (1) \\ 2 & a \div e = q\ (100000)\text{; Quere } a \text{ and } e. \end{array}\right.$$

$$\begin{array}{r|c|l} 1 - e & 3 & a = s - e \\ 2 \times e & 4 & a = eq \\ 3 = 4 & 5 & s - e\,(= a) = eq \\ 5 + e & 6 & s = qe + e \\ 6 \div \overline{q + 1} & 7 & \dfrac{s}{q+1} = e\left(= \dfrac{1}{100001}\right) \\ 3\,,7 & 8 & a = s - \dfrac{s}{q+1} = \dfrac{qs}{q+1}\left(= \dfrac{100000}{100001}\right). \end{array}$$

Question 3.

A Gentleman finding ſeveral Beggars at his Door, gave each of them b (3) Pence, and had c (6) Pence remaining; but he wou'd have given them d (4) Pence each, but that he wanted f (2) Pence for to do ſo. How many Beggars were there?

$$\begin{array}{r|c|l} \text{Suppoſe} & 1 & a = \text{ the Number of Beggars } = ? \\ \text{Then} & 2 & ba + c = da - f, \text{ by the Queſtion.} \\ 2 + f & 3 & ba + c + f = da. \\ 3 - ba & 4 & c + f = da - ba. \\ 4 \div \overline{d - b} & 5 & \dfrac{c + f}{d - b} = a\left(= \dfrac{6 + 2}{4 - 3} = 8 \text{ Beggars}\right). \end{array}$$

Question 4.

One being ask'd how old he was, anſwer'd, if $\frac{c}{b}$ ($\frac{1}{60}$) the Number of Years I liv'd were Multiplyed by $\frac{p}{q}$ ($\frac{5}{8}$) that Number, the Product wou'd be my Age. I demand his Age?

$$\begin{array}{r|c|l} \text{Suppoſe} & 1 & a = \text{his Age.} \\ \text{Then} & 2 & \dfrac{ca}{b} \times \dfrac{pa}{q} = a = \dfrac{cpaa}{bq} \text{ by the Queſtion.} \\ 2 \times \dfrac{bq}{a} & 3 & bq = cpa. \\ 3 \div cp & 4 & \dfrac{bq}{cp} = a\left(= \dfrac{60 \times 8}{1 \times 5} = 96 \text{ Years}\right). \end{array}$$

Question

Question 5.

A Man with his Wife did usually drink out a Vessel of Beer in b (12) Days; and they found, by often Experience, that, the Wife being absent, the Man drank it out in c (20) Days. The Question is in how many Days wou'd the Wife alone drink it out at that Rate?

1. Suppose she wou'd drink it out in a Days.

2. Now, in order to find what Quantity he would drink out in a Days, say

c Days .. 1 Vessel :: a Days .. $\frac{a}{c}$ Vessels the Quantity he wou'd drink out in a Days.

3. Therefore the Quantity (or Number of Vessels) which he and she together would drink out in a Days is $= 1 + \frac{a}{c} = \frac{c+a}{c}$ Vessels.

4. But he and she together did also (by the Question) drink out 1 Vessel in b Days; therefore to find what Quantity they both together did drink out in a Days at this Rate, say

b Days .. 1 Vessel :: a Days .. $\frac{a}{b}$ Vessels.

$$\begin{array}{r|l|l}
3 = 4 & 5 & \frac{c+a}{c} = \frac{a}{b}. \\
5 \times cb & 6 & cb + ba = ca. \\
6 - ba & 7 & cb = ca - ba. \\
7 \div \overline{c-b} & 8 & \frac{cb}{c-b} = a \left(= \frac{20 \times 12}{20 - 12} = 30 \text{ Days.}\right)
\end{array}$$

Question 6.

A Hare, being b (50) Paces, or Leaps of her own, before a Grey-hound, makes r (4) Leaps to the Grey-hound's s (3) Leaps: But m (2) Leaps of the Grey-hound's are as much as n (3) Leaps of the Hare's. How many Leaps must the Gray-hound take to catch the Hare?

1. For

1. For the Number of the Gray-hound's Leaps sought put a.

2. Then $s \,..\, r :: a \,..\, \frac{ra}{s}$ (that is, as the Number of the Gray-hound's Leaps is to those of the Hare's in any time, so are all the Gray-hound's Leaps to all the Hare's Leaps after he began to course her.)

3. And $\frac{ra}{s} + b = \frac{ra+sb}{s}$ = the whole Number of Paces the Hare went.

4. $m \,..\, n :: a \,..\, \frac{ra+sb}{s}$ (that is m is to n as the whole Number of the Gray-hound's Leaps is to the whole Number of the Hare's Leaps.)

$$\begin{array}{r|l|l}
4,\ & 5 & \frac{mra+msb}{s} = na. \\
5 \times s & 6 & mra + msb = sna. \\
6 - mra & 7 & msb = sna - mra. \\
7 \div \overline{sn - mr} & 8 & \frac{msb}{sn-mr} = a \left(= \frac{2 \times 3 \times 50}{3 \times 3 - 2 \times 4} = 300\right.
\end{array}$$

Leaps of the Gray-hound's.)

Or, having proceeded as far as the 3d Step, it is evident that a Leaps of the Gray-hound's must be equal to $\frac{ra+sb}{s}$ Leaps of the Hare's: And, by the latter Part of the Question, m Leaps of the Gray-hound's are $= n$ Leaps of the Hare's; therefore ma Leaps of the Gray-hound's are $= na$ Leaps of the Hare's; consequently (by dividing each Part by m) a Leaps of the Gray-hound's are $= \frac{n}{m}a$ Leaps of the Hare's, which are (by what has been here said) $= \frac{ra+sb}{s}$: And, by Multiplying each Part by m, you get the Equation in the 5th Step foregoing.

[r] *Vitruvius* (in *Lib. 9. Cap. 3.*) reporteth that King *Hiero* having given Commandment for the making a Crown of pure Gold, was inform'd, that the Workman had detained part of the Gold, and mixt the rest with as much Silver as he had stoln of Gold. The King being much displeas'd at the Deceit, recommended

commended the Examination of the Businefs to the famous *Archimedes* of *Syracuſe*, who, without defacing the Crown, diſcover'd the Cheat in this Manner, *viz.* Experience telling him that a Quantity of Gold wou'd poſſeſs leſs Room or Space than a Quantity of Silver of the ſame Weight, and conſequently that a mixt Maſs of Gold and Silver of the ſame Weight wou'd take up ſome mean Space between the two former, He made a Maſs of pure Gold of the ſame Weight with the Crown, likewiſe another Maſs of Silver of the ſame Weight with the Crown; then having put the Crown, as alſo the other two Maſſes ſeverally into a Veſſel filled up to the Brim with Water, he diligently reſerv'd the Water flowing over into another Veſſel; and from thoſe three Quantities of Water ſo expell'd, he found out the Quantity of Gold and Silver in the Crown. But, ſince *Vitruvius* delivers not the Practical Operation, I ſhall here ſhew the ſame after the manner of others.

Queſtion 7.

Suppoſe the Weight of the Crown, as alſo of the two Maſſes to have been each $= b\ (10)$ ℔; alſo that by putting the Maſs of Gold into the Veſſel, $c\ (.52)$ ℔ of Water was thereby expell'd, by putting in the Maſs of Silver, $d\ (.92)$ ℔; and by putting in the Crown, $f\ (.64)$ ℔. The Queſtion is to know how much Gold and how much Silver the Crown was compos'd of?

Suppoſe of a ℔ of Gold, and therefore of $b - a$ ℔ of Silver; then,

If b ℔ of Gold expel c ℔ of Water, a ℔ of Gold will expel $\frac{ca}{b}$ ℔ of Water.

Again, if b ℔ of Silver expel d ℔ of Water, $b - a$ ℔ of Silver will expel $\frac{db - da}{b}$ ℔ of Water.

And therefore the Quantity of Gold and Silver in the Crown will expel $\frac{ca + db - da}{b}$ ℔ of Water, which muſt be $= f$ ℔ of Water, by the Queſtion.

And by Multiplying each Part by b, we have $ca - da + db = bf$.

By Tranſpoſition $ca - da = bf - bd$.

By Diviſion $a = \frac{bf - bd}{c - d} = \frac{d - f}{d - c} b$ ($= 7$ ℔ of Gold).

And therefore $b - a = b - \frac{d - f}{d - c} b$ ($= 3$ ℔ of Silver.)

Queſtion

Question 8.

To divide b (90) into four such Parts that, if the first be increas'd with d (2), the second lessened by d, the third Multiplyed by d, and the fourth Divided by d; the Sum, Remainder, Product and Quotient may be equal between themselves.

Suppose the four Parts sought be equal to a, e, u and y respectively; then

Solution.

	1	$a+e+u+y=b$.
And	2	$a+d=e-d$,
	3	$a+d=du$.
	4	$a+d=\frac{y}{d}$.
	5	$a=a$.
$2+d$	6	$a+2d=e$.
$3\div d$	7	$\frac{a+d}{d}=u$.
$4\times d$	8	$da+dd=y$.
$5+6+7+8=1$	9	$2a+2d+\frac{a+d}{d}+da+dd(=a+e+u+y)=b$.
9 Reduc'd	10	$a=\frac{bd}{dd+2d+1}-d$ (= 18).
10, 6	11	$e=\frac{bd}{dd+2d+1}+d$ (= 22).
10, 7	12	$u=\frac{b}{dd+2d+1}$ (= 10).
10, 8	13	$y=\frac{bdd}{dd+2d+1}$ (= 40).

Question 9.

$$a+\frac{e+x+y+z}{3}=b\ (2380).$$

$$e+\frac{a+x+y+z}{4}=c\ (2205).$$

$$x+\frac{a+e+y+z}{5}=d\ (2268).$$

$$y+\frac{a+e+x+z}{6}=f\ (2450).$$

and

and $z+\frac{a+e+x+y}{7}=g$ (2700). The Question is what is the Value of a?

The Question being reduc'd to Integers gives

	1	$3a+e+x+y+z=3b.$
	2	$4e+a+x+y+z=4c.$
	3	$5x+a+e+y+z=5d.$
	4	$6y+a+e+x+z=6f.$
	5	$7z+a+e+x+y=7g.$
$2-1$	6	$3e-2a=4c-3b.$
$3-1$	7	$4x-2a=5d-3b.$
$4-1$	8	$5y-2a=6f-3b.$
$5-1$	9	$6z-2a=7g-3b.$
$6+2a$	10	$3e=2a+4c-3b.$
$7+2a$	11	$4x=2a+5d-3b.$
$8+2a$	12	$5y=2a+6f-3b.$
$9+2a$	13	$6z=2a+7g-3b.$
	14	$3a=3a.$
$10\div 3$	15	$e=\frac{2a}{3}+\frac{4c-3b}{3}.$
$11\div 4$	16	$x=\frac{2a}{4}+\frac{5d-3b}{4}.$
$12\div 5$	17	$y=\frac{2a}{5}+\frac{6f-3b}{5}.$
$13\div 6$	18	$z=\frac{2a}{6}+\frac{7g-3b}{6}.$
$14+15+16+17+18=1$	19	$3a+e+x+y+z=3a+\frac{2a}{3}+\frac{2a}{4}+\frac{2a}{5}+\frac{2a}{6}+\frac{4c}{3}+\frac{5d}{4}+\frac{6f}{5}+\frac{7g}{6}-\frac{3b}{3}-\frac{3b}{4}-\frac{3b}{5}-\frac{3b}{6}=3b.$
19 Reduc'd	20	$a=\frac{351b-80c-75d-72f-70g}{294}$ (=420).

a being thus found, e, x, y and z will be made known by the 15th, 16th, 17th and 18th Steps (that is $e=840$, $x=1260$, $y=1680$, and $z=2100$).

Question 10.

Two Ships A and B loaden with the same sort of Wine sailing by a Pass, were oblig'd to pay Toll according to the Quantity of Wine they had on Board. A had on Board b (250) Hogsheads, out of which she paid c (1) Hogshead and d (36)

(36) Shillings more. B had f (400) Hogsheads, out of which she paid g (2) Hogsheads, and receiv'd out of them h (20) Shillings. The Question is what the Wine was valued at per Hogshead?

1. Suppose at a Shillings per Hogshead.

2. Then A, having on Board b Hogsheads of Wine, paid (c Hogsheads $+ d$ Shillings or) the Value of $: ca + d:$ Shillings out of them.

3. And B, having on Board f Hogsheads of Wine, paid the Value of $: ga - h:$ Shillings out of them.

4. Now, if b Hogsheads pays $ca + d$ Shillings, f Hogsheads must pay $\frac{cfa + df}{b}$ Shillings $= ga - h$ Shillings.

$4 \times b$	5	$cfa + df = bga - bh.$
$5 - cfa + bh$	6	$df + bh = bga - cfa.$
$6 \div \overline{bg - cf}$	7	$\frac{df + bh}{bg - cf} = a$ (= 94 Shill. = 4*l.* 14*s.* 0*d.* per Hogshead).

Question 11.

A Merchant bought b (40) Bushels of Wheat, c (24) Bushels of Barley, and d (20) Bushels of Oats together for p ($15\frac{3}{5}$) Pounds; afterwards he bought, at the same Price for each Sort of Grain, f (26) Bushels of Wheat, g (30) Bushels of Barley, and h (50) Bushels of Oats together for q (16) Pounds; and after that he bought, at the same Price for each sort of Grain, k (24) Bushels of Wheat, l (120) Bushels of Barley, and m (100) Bushels of Oats together, for r (34) Pounds. The Question is what each sort of Grain cost him per Bushel?

1. Suppose the Wheat cast him a, the Barley e, and the Oats y Pounds per Bushel: then

	2	$\left.\begin{array}{l} ba + ce + dy = p \\ fa + ge + hy = q \\ ka + le + my = r \end{array}\right\}$ By the Question.
	3	
	4	
$2 - ce - dy$	5	$ba = p - ce - dy.$
$5 \div b$	6	$a = \frac{p - ce - dy}{b}.$
$3 - ge - hy$	7	$fa = q - ge - hy.$
$7 \div f$	8	$a = \frac{q - ge - hy}{f}.$
$4 - le - my$	9	$ka = r - le - my.$
$9 \div k$	10	$a = \frac{r - le - my}{k}.$
$6 = 8$	11	$\frac{p - ce - dy}{b} = \frac{q - ge - hy}{f}.$
$11 \times bf$	12	$fp - fce - fdy = bq - bge - bhy.$

$12 - fp + bge + fdy$	13	$bge - fce = bq - fp + fdy - bhy$.
$13 \div \overline{bg - fc}$	14	$e = \dfrac{bq - fp + fdy - bhy}{bg - fc}$.
$8 = 10$	15	$\dfrac{q - ge - hy}{f} = \dfrac{r - le - my}{k}$.
$15 \times fk$	16	$kq - kge - khy = fr - fle - fmy$.
16 Transp.	17	$fle - kge = fr - kq + khy - fmy$.
$17 \div \overline{fl - kg}$	18	$e = \dfrac{fr - kq + khy - fmy}{fl - kg}$.
Put	19	$B = bq - fp$, $C = fd - bh$ and $D = bg - fc$
and	20	$F = fr - kq$, $G = kh - fm$ and $H = fl - kg$.
14, 19	21	$e = \dfrac{B + Cy}{D}$.
18, 20	22	$e = \dfrac{F + Gy}{H}$.
$21 = 22$	23	$\dfrac{B + Cy}{D} = \dfrac{F + Gy}{H}$.
$23 \times DH$	24	$HB + HCy = DF + DGy$.
$24 - HCy - DF$	25	$HB - DF = DGy - HCy$.
$25 \div \overline{DG - HC}$	26	$\dfrac{HB - DF}{DG - HC} = y$ ($\frac{1}{10}$ Pounds = 2 Shillings).

The Value of y being thus found, that of e ($= \frac{3}{20}$ Pounds = 3 Shillings) will be given by the 21st or 22d Step; and then the Value of a ($= \frac{1}{4}$ Pound = 5 Shillings) will be had by the 6th, 8th or 10th Step.

Question 12.

If b (12) Cows eat up c ($3\frac{1}{3}$) Acres of Pasture in d (4) Months, and f (21) Cows eat up g (10) Acres of the same Pasture in h (9) Months. The Question is, supposing the Grass to grow uniformly, how many Cows will eat up k (36) Acres of the said Pasture in l (18) Months?

1. Suppose $y = ?$

2. Suppose the Quantity of Grass upon each Acre of the Pasture at the Time of putting the Cows upon't to be $= z$; then the Quantity of Grass upon c Acres is cz, upon g Acres is gz, and upon k Acres is kz.

3. Suppose the Quantity of Grass produc'd by each Acre of the said Pasture each Month to be $= a$; then the Quantity that will be produc'd by c Acres in d Months is $= acd$, by g Acres in h Months $= agh$, and by k Acres in l Months is $= akl$.

 4. Sup-

4. Suppose the Quantity of Grass that each Cow eats up each Month to be $= e$; then the Quantity of Grass which b Cows will eat up in d Months is $= ebd$, which f Cows will eat up in h Months is $= efh$, and which y Cows will eat up in l Months is $= eyl$.

Then, by the Question,

	5	$ebd = cz + acd.$
	6	$efh = gz + agh.$
	7	$eyl = kz + akl.$
5,	8	$e = \dfrac{cz + acd}{bd}.$
6,	9	$e = \dfrac{gz + agh}{fh}.$
8, 9	10	$a = \dfrac{:fhc - bdg: \times z}{:bg - fc: \times hd}.$
Again, 5,	11	$a = \dfrac{ebd - cz}{cd}.$
6,	12	$a = \dfrac{efh - gz}{gh}.$
11, 12	13	$e = \dfrac{:h - d: \times cgz}{:bg - cf: \times dh}.$
7, 10, 13	14	$y \left(= \dfrac{kz + akl}{el}\right) = \dfrac{:l-d: \times bfck - :l-h: \times dbgk}{:h - d: \times cgl}$ (= 54 Cows).

* *See the* 11*th Prob.* (*in Pag.* 89) *of Sir* If. Newton's *Arith. Universalis.*

Question 13.

Suppose a Clock hath two Indices A and B, and that A is carried b (24) Circumferences (or b times round) in c (24) Hours; and B is carried, the same Way, d (1) Circumferences in f (12) Hours. The Question is, if they were conjoin'd at this Instant, in how many Hours wou'd they be conjoin'd again?

Suppose $a =$ Number of Hours required,

Then c Hours .. b Circumferences :: a Hours .. $\dfrac{ba}{c}$ Circumferences $=$ A's whole Course in a Hours.

f Hours .. d Circumferences :: a Hours .. $\dfrac{da}{f}$ Circumferences $=$ B's whole Course in a Hours.

It

It is evident that the Index whose Motion is swiftest will out go the slowest 1 Circumference in a Hours; whence $\frac{ba}{c} \backsim \frac{da}{f} = 1$; that is $\frac{bfa \backsim dca}{cf} = 1$.

Wherefore, Multiplying each Part by cf,

$$bfa \backsim dca = cf.$$

And, by Dividing each part by $bf \backsim dc$, you'll have $a = \frac{cf}{bf \backsim dc}$ $\left(= \frac{24 \times 12}{24 \times 12 - 1 \times 24} = 1\tfrac{1}{11}\right.$ Hours.)

Lemma.

To find the * least Number, which being Divided by two or more Numbers shall have no Remainder.

Rule.

Reduce the two first taken Numbers to a common Denominator; then Multiply their Numerators by one another, and Multiply the greatest common Measure of the Numerators by their common Denominator; by the last Product Divide the Product of the Numerators, and the Quotient is the least Number that can be divided by the two first taken Numbers without having a Remainder; which found Number take as one, and one of the remaining Numbers (if there be any) as an other, and work with these in like manner, as with the two first taken Numbers; and so proceed with all the other Numbers (if more there be), and the last Quotient is the Number required.

In Order to Demonstrate this *Lemma*, I will suppose the two first taken Numbers, after they are reduc'd to a common Denominator, to be equal to $\frac{b}{c}$ and $\frac{d}{c}$, and g to be equal to the greatest common Measure of b and d, and a to be equal to the least Number that can be Divided by $\frac{b}{c}$ and $\frac{d}{c}$ without leaving a Remainder. I say, that a is $= \frac{bd}{cg}$.

---

* N.B. *The Numbers here treated of are suppos'd to be Affirmative.*

Demon-

Demonstration.

For, if $\frac{bd}{cg}$ be Divided by $\frac{b}{c}$ and $\frac{d}{c}$ severally, the Quotients will be equal to whole Numbers, to wit to $\frac{d}{g}$ and $\frac{b}{g}$; wherefore, if a be not $\sqsupset \frac{bd}{cg}$, it is equal to it. Now, in order to demonstrate that a is not $\sqsupset \frac{bd}{cg}$, suppose $a \sqsupset \frac{bd}{cg}$; and suppose e to be such that it may be equal to any Number that is $\sqsubset 1$; then any Number whatsoever, which is $\sqsupset \frac{bd}{cg}$ may be design'd by $\frac{bd}{cge}$: And suppose $a = \frac{bd}{cge}$. Now this Value of a being Divided by $\frac{b}{c}$ and $\frac{d}{c}$ severally gives the two Quotients equal to $\frac{d}{ge}$ and $\frac{b}{ge}$, either or both of which Quotients is or are equal to a Fractional Number or Fractional Numbers, which is contrary to what is required, or g is not equal to the greatest common Measure of b and d but ge, which is contrary to our Supposition; wherefore the Supposition, namely, that a is $\sqsupset \frac{bd}{cg}$ is absurd: And consequently a is $= \frac{bd}{cg}$.

Hence 'tis evident that all the Numbers which can be Divided by the Values of $\frac{b}{c}$ and $\frac{d}{c}$ without having a Remainder are $\frac{bd}{cg}$ and its Multiples, *viz.* $\frac{bd}{cg}$, $\frac{2bd}{cg}$, $\frac{3bd}{cg}$, $\frac{4bd}{cg}$, &c. so that if you take one (if there be any) of the remaining Numbers, and find the least Number that can be measur'd by it and the Value of the before found Quantity $\frac{bd}{cg}$, you will have the least Number that can be measur'd (or Divided without leaving a Remainder) by the Values of $\frac{b}{c}$, $\frac{d}{c}$ and by the last taken Number. Finally by proceeding so with all the other Numbers (if there remains more) you will manifestly find the least Number that can be Divided by the given Numbers without leaving any Remainder. *Q. E. D.*

Question

Question 14.

Suppose a Clock hath four Indices, to wit A, B, C and D. A is carried 1 time round the whole Circumference of the Clock in 1 Hour; B is carried the same way $1\frac{1}{2}$ times round in 1 Hour; C is carried the same way $2\frac{1}{3}$ times round in 1 Hour; and D is carried the same way 3 times round in 1 Hour. The Question is, in how many Hours, after all the Indices commence a Motion from the same Point, will they all be conjoin'd again.

Suppose A to be conjoin'd with $\left\{\begin{matrix} B \\ C \\ D \end{matrix}\right\}$ in $\left\{\begin{matrix} a \\ e \\ y \end{matrix}\right\}$ Hours.

Since A's Motion is the slowest, it is evident that either of the rest, to wit, B, C or D will be carried 1 Circumference more than A at the time that it will be conjoin'd with, or overtake A: Wherefore, in order to find in how many Hours A and B will be conjoin'd, say

$a \times 1 + 1$ Circumferences $= a \times 1\frac{1}{2}$ Circumferences,

That is $a + 1 = 1\frac{1}{2}a$; therefore $1 = 1\frac{1}{2}a - a = \frac{1}{2}a$; Consequently $2 = a$; that is to say A and B will be conjoin'd in 2 Hours.

In like manner e will be found $= \frac{3}{4}$, which is the Number of Hours wherein the Conjunction of A and C will be accomplish'd.

And by a like Process with the foregoing in finding the Value of a, the Value of y will be found to be $\frac{1}{2}$.

Since (as we have now prov'd) A will be conjoin'd with B in 2 Hours, after it was conjoin'd with it before, it is manifest that they will likewise be conjoin'd in every Multiple of 2 Hours, as 4, 6, 8, 10, &c. Hours. In like manner it will appear that A will be conjoin'd with C in $\frac{3}{4}$ Hours and every Multiple thereof, as $\frac{3}{4}$, $\frac{6}{4}$, $\frac{9}{4}$, $\frac{12}{4}$, $\frac{15}{4}$, &c. Hours; and that A and D will be conjoin in $\frac{1}{2}$ Hour and every Multiple of $\frac{1}{2}$ Hour, as $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$. &c. Hours; wherefore the least Number that can be Divided without having a Remainder by 2, $\frac{3}{4}$ and $\frac{1}{2}$ is the Number of Hours wherein the Conjunction of A with B, with C and with D will be accomplish'd; and, of consequence, A, B, C and D will then be on the same Point.

Now, in order to find the least Number that can be Divided by 2, $\frac{3}{4}$ and $\frac{1}{2}$ without leaving a Remainder, I proceed, as my Rule directs, thus:

The two first taken Numbers, suppose 2 and $\frac{3}{4}$, being reduc'd to a common Denominator, are equal to $\frac{8}{4}$ and $\frac{3}{4}$ respectively; and

and the least Number that can be Divided by $\frac{8}{4}$ and $\frac{3}{4}$ without a Remainder will be found (by the said Rule) $= \frac{8 \times 3}{4 \times 1} = \frac{24}{4}$: And the least Number that can be Divided by $\frac{24}{4}$ and $\frac{1}{2}$ $(\frac{2}{4})$ without a Remainder will be also found (by our Rule, or *Lemma*) $= \frac{24 \times 2}{4 \times 2} = 6$; wherefore 6 is the Number of Hours wherein A, B, C and D, after their being mov'd from the same Point, will be conjoin'd; that is to say 6 Hours is the Answer required.

Question 15.

A Clock hath three Indices A, B and C. A is carried $\frac{b}{p}$ $(\frac{3}{2})$ times round the whole Circumference of the Clock in one Hour. B is carried the same way $\frac{c}{q}$ $(\frac{11}{5})$ times round in one Hour. And C is carried the same way $\frac{d}{r}$ $(\frac{17}{7})$ times round in one Hour. The Question is, supposing them to be mov'd at once from the same Point of the Circumference, as also supposing $\frac{b}{p} \sqsupset \frac{c}{q}$ and $\frac{c}{q} \sqsupset \frac{d}{r}$, in how many Hours from the said Moving will they all three be conjoin'd again.

Suppose A to be conjoin'd with $\left\{ \begin{matrix} B \\ C \end{matrix} \right\}$ in $\left\{ \begin{matrix} a \\ e \end{matrix} \right\}$ Hours.

Since A's Motion is the slowest, it is evident that either of the rest, to wit B or C will be carried 1 Circumference more than A at the time that it will overtake, or be conjoin'd with A; wherefore $\frac{ab}{p} + 1$ Circumferences $= \frac{ac}{q}$ Circumferences; therefore $abq + pq = acp$,

And, by Transposition and Division $\frac{pq}{cp - bq} = a$.

In like manner e will be found $= \frac{pr}{dp - br}$;

Wherefore A will be conjoin'd with B in $\frac{pq}{cp - bq}$ Hours and every Multiple thereof, as $\frac{pq}{cp - bq}, \frac{2pq}{cp - bq}, \frac{3pq}{cp - bq}$ &c.

&c. Hours; and A will be conjoin'd with C in $\frac{pr}{dp-br}$ Hours and every Multiple thereof, *viz.* in $\frac{pr}{dp-br}$, $\frac{2pr}{dp-br}$, $\frac{3pr}{dp-br}$, &c. Hours; and therefore A will be conjoin'd with B, as also with C, and consequently B will be conjoin'd with C in every Quantity that can be Divided by $\frac{pq}{cp-bq}$ and $\frac{pr}{dp-br}$ without leaving a Remainder: Consequently the least Quantity that can be Divided by $\frac{pq}{cp-bq}$ and $\frac{pr}{dp-br}$ without leaving a Remainder is = the Number of Hours wherein the Conjunction of A, B and C will be accomplish'd, or the Answer required.

Now $\frac{pq}{cp-bq}$ and $\frac{pr}{dp-br}$ being reduc'd to a common Denominator are $\frac{dppq-bpqr}{cdpp-bcpr-bdpq+bbqr}$ and $\frac{cppr-bpqr}{cdpp-bcpr-bdpq+bbqr}$, and the least Quantity that can be Divided by those without leaving a Remainder will be found, by our *Lemma* (putting w = the greatest common Measure of $dpq-brq$ and $cpr-bqr$) to be $\frac{cdp^3qr-bcppqrr-bdppqqr+bbpqqrr}{:cdpp-bcpr-bdpq+bbqr:\times w}=\frac{pqr}{w}$ $\left(=\frac{2\times5\times7}{1}=70\right)$ Hours. Answer.

Lemma.

If b, c and d be each of them equal to known Affirmative whole Numbers; and $ba+c=de$: 'Tis required to find the least Values of a and e in Affirmative whole Numbers.

Rule.

From * d, or (if c be $\sqsubset d$) the * least Multiple of d that is not less than * c Subtract c, and, if there be a Remainder, Divide it by b (But, Note, when the Remainder is $\sqsupset b$, this Division is superfluous); if any thing remains, add to it d, or the least Multiple of d, so as the Sum may not be less than b, which Sum Divide by b; and if any thing yet remains, add to it d, or the least Multiple of d, so as the Sum may not be less than b, which Sum Divide by b: Proceed thus 'till nothing remains (or 'till you have the same Remainder twice, in which Case the Question pro-

* *i. e. the Values of*

pos'd is manifestly impossible) then, putting $z =$ Number of all the d's in the Operation, I say that e is $= z$.

For, since e, by *Hyp.* must be equal to a whole Number, de will be equal d or some Multiple of d, as d, $2d$, $3d$, &c. and the least of these, from which c being Subtracted and the Remainder Divided by b will leave no Remainder, is (every whole being equal to all its Parts taken together) *manifestly $= dz$; wherefore $\frac{dz - c}{b}$ is $=$ the smallest Number capable of being an Affirmative whole one: But, by *Hyp.* $\frac{de - c}{b} = a$ must be $=$ the smallest Number capable of being an Affirmative whole one; consequently $\frac{dz - c}{b} = \frac{de - c}{b}$; and $z = e$. Q.E.D.

* *See the following Examples.*

Examples.

1. If $17a + 5 = 19e$, 'tis required to find the least Values of e and a in Affirmative whole Numbers.

Here $17 = b$, $5 = c$ and $19 = d$.

Operation.

$19 = d$	3	9
$c = 5$	$19 = d$	$19 = d$
14	17)22(1	17)28(1
$19 = d$	5	11
17)33(1	$19 = d$	$19 = d$
16	17)24(1	17)30(1
$19 = d$	7	13
17)35(2	$19 = d$	$19 = d$
1	17)26(1	17)32(1
$19 = d$	9	15
17)20(1		$19 = d$
3		17)34(2
		0 Remaider

sought; consequently 11 (= the Number of d's in the Operation) is $= e$; wherefore $19 \times 11 = de = 209$; and $\frac{209 - 5}{17} = 12 = a$.

2. If $6a + 2 = 5e$, 'tis required to find the least Values of e and a in Affirmative whole Numbers.

Here $6 = b$, $2 = c$ and $5 = d$.

Opera-

Operation.

$5 = d$
$c = \underline{2}$
3
$\underline{5} = d$
$6)\underline{8}(1$
2
$\underline{5} = d$
$6)\underline{7}(1$
1
$\underline{5} = d$
$6)\underline{6}(1$
0 Remainder fought: Confequently 4 (= Number of the d's in the Operation) $= e$; wherefore $5 \times 4 = 20 = de$, and $a = \frac{20 - 2}{6} = 3$.

3. But, if $4a + 7 = 14e$, then
$14 = d$
$c = \underline{7}$
$b = 4)\underline{7}(1$
3
$\underline{14} = d$
$4)\underline{17}(4$
1
$\underline{14} = d$
$4)\underline{15}(3$
3 the fame with a foregoing Remainder; and confequently it is impoffible to find the Values of e and a in whole Numbers.

4. If $3a + 16 = 8e$, Quere e and a in the leaft Affirmative Integers.

Here $3 = b$, $16 = c$ and $8 = d$; and (c being $\sqsubset d$) the leaft Multiple of d not lefs than c is $= 16 = 2d$.

Operation.

$$\begin{array}{r} 16 = 2d \\ c = \underline{16} \\ 0 \end{array}$$

Remainder sought; consequently 2 (= Number of the d's in the Operation) $= e$; and $\frac{8 \times 2 - 16}{3} = 0 = a$.

5. If $40a + 16 = 6e$, Quere e and a in the least Affirmative Integers.

Here $40 = b$, $16 = c$ and $6 = d$; and (c being $\sqsupset d$) the least Multiple of d not less than c is $= 18 = 3d$.

Operation.

$$\begin{array}{r} 18 = 3d \\ c = \underline{16} \\ 2 \\ \underline{42 = 7d} \\ 40)\underline{44}(1 \\ 4 \\ \underline{36 = 6d} \\ 40)\underline{40}(1 \\ 0 \end{array}$$

Remainder sought: Consequently $3 + 7 + 6 = 16$ (= the Number of d's in the Operation) is $= e$: Wherefore $6 \times 16 = 96 = de$, and $\frac{96 - 16}{40} (= \frac{de - c}{b}) = 2 = a$.

Scholium.

The least Value of de ($= ba + c$) in Affirmative Integers being thus had, all the other Affirmative Integer Values of de may be found by adding to that least Value the least Affirmative Number that b and d Measures, and the several Multiples of that Number: Or any of the Aff: Values of de being known, and Divided by the aforesaid Number, will give the Remaider = the least Aff: Value of de.

Question 16.

'Tis required to find the least Number, which being Divided by 2, 3, 4 and 5 shall leave 1, 2, 3 and 0 respectively for Remainders.

First, in order to find the least Number, which being Divided by 2 and 3, will leave 1 and 2 respectively for Remainders, I proceed thus, by our *Lemma*.

$$3a + 2 = 2e + 1,$$

Therefore $3a + 1 = 2e$.

Here

Here $3 = b$, $1 = c$ and $2 = d$.

$$\begin{array}{r} 2 = d \\ c = 1 \\ \hline 1 \\ 2 = d \\ \hline 3)3(1 \\ \hline 0 \end{array}$$

0; whence 2 (= the Number of d's) is $= e$; and consequently $2e + 1 = 3a + 2 = 5$ is the least Number, which being Divided by 2 and 3 will leave 1 and 2 respectively: And the least Number that 2 and 3 Measures is (by *Lemma* to *Quest.* 14.) $2 \times 3 = 6$; consequently 5, $5 + 6$, $5 + 6 \times 2$, $5 + 6 \times 3$, &c. equal to 5, 11, 17, 23, &c. are the only Numbers, which being Divided by 2 and 3, will leave 1 and 2 respectively for Remainders; wherefore the least of those Numbers, which being Divided by 4 will leave 3 is the least Number, which being Divided by 2, 3 and 4 will leave 1, 2 and 3 respectively; in order to find which, I proceed thus, by our *Lemma*.

$$6a + 5 = 4e + 3$$

Therefore $6a + 2 = 4e$. Here $6 = b$, $2 = c$ and $4 = d$.

$$\begin{array}{r} 4 = d \\ c = 2 \\ \hline 2 \\ 4 = d \\ \hline 6)6(1 \\ \hline 0 \end{array}$$

0; whence 2 (= the Number of d's) is $= e$; consequently $4e + 3 = 6a + 5 = 11$: And the least Aff: Number, that can be Measur'd by 2, 3 and 4, is 12: Consequently 11, $11 + 12$, $11 + 12 \times 2$, $11 + 12 \times 3$, &c. are the only Numbers, which being Divided by 2, 3 and 4 will leave 1, 2 and 3 respectively for Remainders; wherefore the least of the said Numbers, which being Divided by 5 will leave no Remainder is the Number required; which to find, I proceed thus, by our *Lemma*.

$12a + 11 = 5e + 0$. Here $12 = b$, $11 = c$ and $5 = d$.

$$\begin{array}{r} 15 = 3d \\ c = 11 \\ \hline 4 \\ 10 = 2d \\ \hline 12)14(1. \\ \hline \end{array}$$

$$\begin{array}{l} \quad 2 \\ \quad 10 = 2d \\ 12)12(1 \\ \quad 0 \end{array}$$

; wherefore $3 + 2 + 2 = 7$ (= the Number of d's) is $= e$; and $5e = 12a + 11 = 35$ is the Number required.

Sect. 2. Of Unlimitted Questions.

I will now proceed to give an Example or two of the Method used in arguing about unlimitted Questions, *viz.* such Questions as admit of innumerable Answers, such as those in Alligation Alternate.

Example 1. *Question* 17.

A Tobacconist hath three sorts of Tobacco, *viz.* one of 2*s.* 8*d.* the Pound, another of 20 *d.* the Pound, and a third sort of 16 *d.* the Pound: Of these he wou'd make a Mixture to contain 56 Pounds that may be sold for 22 *d.* the Pound. How much of each sort must he take?

Let $a =$ the Quantity of that worth 2*s.* 8*d.* or 32 *d.* the Pound, $e =$ that of 20 *d.* the Pound, and $y =$ that of 16 *d.* the Pound:

Then $a + e + y = 56$

And $32a + 20e + 16y = 56 \times 22 = 1232.$ } { viz. *each Quantity Multiplyed into its own Price equals their Sum Multiplyed into the mean Price.*

This Question being thus stated it appears, by Part VII. Rule I. that it is capable of innumerable Answers; because for any one of these three Letters a, e, y, there may be taken any Number at pleasure within certain Limits, which are to be discover'd in the following Manner, and all the proper or possible Answers in whole Numbers be thence found, thus,

Let	1	$a + e + y = 56$ } as above.
And	2	$32a + 20e + 16y = 1232$ }
$1 - a$	3	$e + y = 56 - a$
$2 - 32a$	4	$20e + 16y = 1232 - 32a$
3×16	5	$16e + 16y = 896 - 16a$
$4 - 5$	6	$4e = 336 - 16a$
$6 \div 4$	7	$e = 84 - 4a$. Hence $a \sqsupset 84 \div 4\ (21)$.
$3 - 7$	8	$y = 3a - 28$. Hence $a \sqsubset \frac{28}{3}\ (9\frac{1}{3})$

From the two last Steps it appears, that the Quantity signified by a, ought to be less than 21, and greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21 may be taken for the Value of a: Consequently there may be eleven Answers to this Question in whole Numbers.

Suppose

Suppose $a = 10$; then $e = 84 - 40 = 44$, *per* 7th Step, and $y = 30 - 28 = 2$, *per* 8th Step.

Again, if $a = 11$; then $e = 84 - 44 = 40$, *per* 7th Step, and $y = 33 - 28 = 5$, *per* 8th Step. And so on for the Rest, which will be as in the following Table.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it it will be easy to find out and collect all the limitted Answers to any Question of this kind, wherein there are only three Quantities propos'd to be mix'd: But when there are more than three, then the Work requires a little more Trouble; because the single Limits of all the Quantities above two must be found; that is, if there are four Quantities concern'd in the Question, the Limits of two of them must be found; if five Quantities are concern'd, then the Limits of three of them must be found, &c. as in the following Question.

Example 2. *Question* 18.

Suppose it were required to mix four sorts of Wine together, *viz.* one sort worth 7 *s.* 4 *d.* the Gallon, another sort worth 4*s.* 7*d.* the Gallon, a third sort worth 3 *s.* 8 *d.* the Gallon, and a fourth sort worth 2 *s.* 9 *d.* the Gallon. How many of each sort may be taken to make a Mixture of 63 Gallons, so as the whole Quantity may be sold for 5 *s.* 6 *d.* the Gallon, without Loss, &c.

First let all the several Rates, and the mean Rate be reduc'd to one Denomination, *viz.* into Pence.

Viz. 7 *s.* 4 *d.* = 88 *d*; 4*s.* 7*d.* = 55 *d*; 3 *s.* 8 *d.* = 44*d*; 2*s.* 9*d.* = 33*d*; and 5*s.* 6*d.* = 66 *d.* Then put a = the Quantity of that worth 88*d.* the Gallon; e = that of 55*d.* the Gallon; y = that of 44*d.* the Gallon; and u = that of 33*d.* the Gallon. Then

	1	$a + e + y + u = 63$, by the Question.
and	2	$88a + 55e + 44y + 33u = 63 \times 66 = 4158$.
$1 - a$	3	$e + y + u = 63 - a$.
$2 - 88a$	4	$55e + 44y + 33u = 4158 - 88a$.
3×33	5	$33e + 33y + 33u = 2079 - 33a$.
$4 - 5$	6	$22e + 11y = 2079 - 55a$.
$6 \div 11$	7	$2e + y = 189 - 5a$. Hence a is $\sqsupset \frac{189}{5}$ $(37\frac{4}{5})$.
3×55	8	$55e + 55y + 55u = 3465 - 55a$.
$8 - 4$	9	$11y + 22u = 33a - 693$.
$9 \div 11$	10	$y + 2u = 3a - 63$. Hence $a \sqsubset 63 \div 3$ (21).

From

From the 7th and 10th Steps it appears, that the Quantity of that sort of Wine denoted by a must be less then $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: That is it may be $a =$ any Number of Gallons betwixt 21 and $37\frac{4}{5}$.

Next to find the Limits of e, y and u.

Suppose	11	$a = 22$, then will $5a = 110$, and $3a = 66$.
But	12	$2e + y = 189 - 5a = 79$, *per* 7th Step.
$12 - 2e$	13	$y = 79 - 2e$. Hence $e \sqsupset \frac{79}{2} (39\frac{1}{2})$.
Again	14	$e + y + u = 63 - a = 41$, *per* 3d Step.
$14 - e$	15	$y + u = 41 - e$.
$15 - 13$	16	$u = e - 38$. Hence $e \sqsubset 38$.

From the 13th and 16th Steps it appears that if $a = 22$, then $e = 39$, $y = 79 - 2e = 1$, and $u = e - 38 = 1$.

Again,

Suppose	17	$a = 23$, then $5a = 115$, and $3a = 69$.
But	18	$2e + y = 189 - 5a = 74$, *per* 7th Step.
$18 - 2e$	19	$y = 74 - 2e$. Hence $e \sqsupset 74 \div 2 (37)$.
Again	20	$e + y + u = 63 - a = 40$, *per* 3d Step.
$20 - e$	21	$y + u = 40 - e$.
$21 - 19$	22	$u = e - 34$. Hence $e \sqsubset 34$.

From the 19th and 22d Steps it appears that if $a = 23$, then e may be 35 or 36.

Once more for a further Illustration.

Let	23	$a = 24$; then $5a = 120$, and $3a = 72$.
But	24	$2e + y = 189 - 5a = 69$, *per* 7th Step.
$24 - 2e$	25	$y = 69 - 2e$. Hence $e \sqsupset \frac{69}{2} (34\frac{1}{2})$.
Again	26	$e + y + u = 63 - a = 39$, *per* 3d Step.
$26 - e$	27	$y + u = 39 - e$.
$27 - 25$	28	$u = e - 30$. Hence $e \sqsubset 30$.

From hence it appears that if $a = 24$, then e may be either 31, 32, 33 or 34, *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th and 28th Steps: From whence the Values of y and u may be easily found;

$$\text{That is, if} \left\{ \begin{array}{l} e = 31 \\ e = 32 \\ e = 33 \\ e = 34 \end{array} \right\} \text{Then} \left\{ \begin{array}{l} y = 7 \\ y = 5 \\ y = 3 \\ y = 1 \end{array} \right\} \text{And} \left\{ \begin{array}{l} u = 1 \\ u = 2 \\ u = 3 \\ u = 4 \end{array} \right\}.$$

Proceed-

Proceeding thus with all the other Values of a; there may be above 120 Anſwers found to this Queſtion in whole Numbers, and if you pleaſe to put $a =$ to Fractions, there may be found an innumerable Sett of Anſwers.

Thoſe two Examples being well underſtood (eſpecially if the laſt be thoroughly purſued) may ſuffice to ſhew the Method of Limitting Anſwers to all ſorts of Queſtions of this kind.

Sect. 3. Queſtions producing Simple Quadratick, &c. Equations.

Queſtion 19.

A certain Footman A departed from *London* towards *Lincoln*; and at the ſame time another Footman B departed from *Lincoln* towards *London*, each keeping the ſame Road. When they met A ſaid to B I find I have travell'd b (20) Miles more than you, and have gone as many Miles in c ($6\frac{2}{3}$) Days, as you have gone Miles in all hitherto. But ſaid B at the end of d (15) Days hence I ſhall be at *London*, if I travel ſtill after the ſame Rate. *Quere* the Diſtance of theſe two Cities from one another, and how many Miles each Footman had travell'd when they met.

1. Suppoſe the Number of Miles that A went each Day to be $= a$, the Number of Days that he was on the Road to the time that he met B $= z$; then the Number of Miles that he went to the time he met B will be $= za$.

2. Suppoſe the Number of Miles that B went each Day $= e$; then the Number of Days that B was on the Road to the time that he met A is (ſince B parted from *Lincoln* at the ſame time that A parted from *London*) $= z$; and the Number of Miles that he went to the time he met A will be $= ze$.

	3	$za = ze + b$ } *per* Queſtion.
	4	$ca = ze$
	5	$de = za$
$4 \div c$	6	$a = \frac{ze}{c}$.
$5 \div z$	7	$a = \frac{de}{z}$.
$6 = 7$	8	$\frac{ze}{c} = \frac{de}{z}$.
8,	9	$z = \sqrt{cd}\ (= \sqrt{6\frac{2}{3} \times 15} = \sqrt{100} = 10)$ Days.
$3 \div z$	10	$a = \frac{ze + b}{z}$.
$7 = 10$	11	$\frac{de}{z} = \frac{ze + b}{z}$.

$11 \times z$	12	$de = ze + b$.
9, 12	13	$de = e\sqrt{cd} + b$.
$13 - e\sqrt{cd}$	14	$de - e\sqrt{cd} = b$.
$14 \div \overline{d - \sqrt{cd}}$	15	$e = \frac{b}{d - \sqrt{cd}} (= \frac{20}{15 - 10} = 4)$ Miles.
6, 9, 15	16	$a = \frac{b}{\sqrt{cd} - c} (= \frac{20}{10 - 6\frac{2}{3}} = 6)$ Miles.
9, 16, 15	17	$za + ze = \frac{b}{1 - \sqrt{\frac{c}{d}}} + \frac{b}{\sqrt{\frac{d}{c}} - 1}$ (= 60

$+ 40 = 100) =$ the Number of Miles that A and B went when they met; that is = the Distance in Miles from *London* to *Lincoln*.

Question 20.

It is is required to find three such Numbers, that the Sum of the First and Second Multiplyed by the Third, may be $= b$ (63); and the Sum of the Second and Third Multiplyed by the First, may be $= c$ (28); also the Sum of the First and Third being Multiplyed by the Second may be $= d$ (55).

Let	1	a, e and y reqresent the First, Second and Third Numbers sought.
Then	2	$ay + ey = b$ ⎫
	3	$ea + ya = c$ ⎬ *per* Question.
	4	$ae + ye = d$ ⎭
$2 + 3 + 4$	5	$2ay + 2ey + 2ae = b + c + d$.
Let	6	$b + c + d = s$.
5, 6	7	$ay + ey + ae = \frac{s}{2}$.
$7 - 4$	8	$ay = \frac{1}{2}s - d$.
$8 \div a$	9	$y = \frac{s - 2d}{2a}$.
$7 - 3$	10	$ey = \frac{1}{2}s - c$.
$7 - 2$	11	$ae = \frac{1}{2}s - b$.
$11 \div a$	12	$e = \frac{s - 2b}{2a}$.
9×11	13	$ey = \frac{ss - 2ds - 2bs + 4bd}{4aa}$.
$10 = 13$	14	$\frac{1}{2}s - c = \frac{ss - 2ds - 2bs + 4bd}{4aa}$.
$14 \times 4aa$	15	$2saa - 4caa = ss - 2ds - 2bs + 4bd$.
$15 \div \overline{2s - 4c}$	16	$aa = \frac{ss - 2ds - 2bs + 4bd}{2s - 4c}$.

16 ww 2	17	$a = \sqrt{\dfrac{ss - 2ds - 2bs + 4bd}{2s - 4c}}$ (= 2).
Then 12,	18	$e = \dfrac{s - 2b}{2a}$ (= 5).
And 9,	19	$y = \dfrac{s - 2d}{2a}$ (= 9).

Question 21.

There is a Wall containing b (18225) Cubical Feet. The Height is c (5) times the Breadth; the Length is d (8) times the Height. I demand the Breadth of the Wall.

Suppose	1	a = the Number of Feet in the Breadth of the Wall.	
Then	2	ca is = the Numb. of Feet in its Height	By the Question.
	3	dca = the Numb. of Feet in its Length	
and	4	$a \times ca \times dca = dcca^3 = b$.	
$4 \div dcc$	5	$a^3 = \dfrac{b}{dcc}$.	
5 ww 3	6	$a = \sqrt[3]{\dfrac{b}{dcc}}$ $\left(= \sqrt[3]{\dfrac{18225}{8 \times 5 \times 5}} = \sqrt[3]{91.125} = 4.5\right.$ Foot).	

CHAP. II.

Questions producing Affected Quadratick, &c. Equations.

A B C D, &c.

Lemma. |—|—|—|—

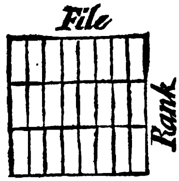
IF along with the one thing A, in any straight Line, be plac'd any Number of Things B, C, D, &c. at any Distance or Distances from one another; I say the Number of the things A, B, C, D, &c. is by one more than the Number of Intervals AB, BC, CD, &c.

Demonstration.

If in the said straight Line along with, and at any Distances from the one thing A, you place n things B, C, D, &c. (n being equal to any whole Number whatsoever) it is evident that you place n Intervals AB, BC, CD, &c. Consequently the Number of things is $n + 1$, and the Number of Intervals is n. *Q.E.D.*

Question 1.

b (36) Soldiers are to be plac'd in a Square Battle in ſuch manner, that every two Soldiers next to one another in Rank may have *c* (8) Foot Diſtance between the Centers of their Stations; and every two Soldiers next to one another in File may have *d* (3) Foot Diſtance between the Centers of their Stations. How many Soldiers muſt be plac'd in Rank, and how many in File?



1. Suppoſe $a =$ the Number of Feet in the Side of the Square.
2. Then (by the Nature of the Queſtion, and by the foregoing *Lemma*). $\frac{a}{c} + 1 = \frac{a+c}{c}$ is $=$ the Number in Rank.
3. And $\frac{a}{d} + 1 = \frac{a+d}{d}$ is $=$ the Number in File.
4. Therefore $\frac{aa + :c+d: \times a + cd}{cd} = b$.

$4 \times cd$	5	$aa + :c+d: \times a + cd = bcd.$
$5 - cd$	6	$aa + :c+d: \times a = bcd - cd.$
Suppoſe	7	$c+d = g$, and $c-d = h$.
6, 7	8	$aa + ga = bcd - cd.$
Comp. □	9	$aa + ga + \frac{1}{4}gg = bcd - cd + \frac{1}{4}gg = bcd + \frac{1}{4}hh.$
9 ɯ 2.	10	$a + \frac{1}{2}g = \sqrt{:bcd + \frac{1}{4}hh:}$
$10 - \frac{1}{2}g$	11	$a = \sqrt{:bcd + \frac{1}{4}hh:} - \frac{1}{2}g$ (= 24).
2, 7, 11	12	$\frac{\sqrt{:bcd + \frac{1}{4}hh:} + \frac{1}{2}h}{c}$ is = the Number in Rank (= 4).
3, 7, 11	13	$\frac{\sqrt{:bcd + \frac{1}{4}hh:} - \frac{1}{2}h}{d}$ is = the Number in File (= 9).

Question 2.

The Sum (*b*) of the Squares of two Numbers being given, as alſo (*c*) the double Product of the Multiplication of the ſame two Numbers; to find the Numbers ſeverally.

Suppose	1	$a =$ the greater Number;
and	2	$e =$ the lesser.
Then	3	$aa + ee = b$ } *per* Question.
and	4	$2ae = c$ }
$4 \div 2a$	5	$e = \frac{c}{2a}.$
5 [illegible] 2	6	$ee = \frac{cc}{4aa}.$
3, 6	7	$aa + \frac{cc}{4aa} = b.$
$7 \times a^2$	8	$a^4 + \frac{cc}{4} = baa.$
8 Transp.	9	$a^4 - baa = -\frac{cc}{4}.$
Comp. □	10	$a^4 - baa + \frac{bb}{4} = \frac{bb - cc}{4}.$
10 ɯ 2.	11	$aa - \frac{b}{2} = \pm\sqrt{\frac{bb - cc}{4}}.$
$11 + \frac{b}{2}$	12	$aa = \frac{b \pm \sqrt{:bb - cc:}}{2}.$
12 ɯ 2.	13	$a = \sqrt{\frac{b \pm \sqrt{:bb - cc:}}{2}}.$
$3 - ee$	14	$aa = b - ee.$
$12 = 14$	15	$\frac{b \pm \sqrt{:bb - cc:}}{2} = b - ee.$
15 Transp.	16	$ee = \frac{b \mp \sqrt{:bb - cc:}}{2}.$
16 ɯ 2.	17	$e = \sqrt{\frac{b \mp \sqrt{:bb - cc:}}{2}}.$

Scholium.

From the 13th and 17th Steps arises this *Canon* for Extracting the Square-Root of Binomial or Residual Numbers.

Canon.

From the Square of the given Sum of the Squares (or from the Square of the greater Part of the given Binomial or Residual) Subtract the Square of the double Product given (or Subtract the Square of the lesser Part); then Add and Subtract the Square-Root of the Remainder to and from the given Sum of the Squares (or

to and from the said greater Part). Lastly connect the Square-Roots of the half of that Sum and Remainder by the Sign +, if a Binomial be propos'd, but rightly by —, if a Residual; so you'll have the desired Square-Root.

Example.

Let it be required to Extract the Square-Root of $27 + \sqrt{704}$.

Operation.

From the Square of the greater Part 27, *viz.* . . . 729
Subtract the Square of the lesser Part $\sqrt{704}$, *viz.* . . . 704

And the Remainder is 25
The Square-Root of which Remainder is 5
Which added to the greater Part 27 is 32
The half of which Sum is 16
The Square-Root of which is the greater Number = 4
The Square-Root 5 Subtracted from the greater Part 27 leaves 22
The half of which is 11
And the Square-Root thereof is the lesser Number = $\sqrt{11}$
I say the Root sought shall be the two Parts so found connected by +, that is $4 + \sqrt{11}$:
But if — insteed of + be prefixt to the lesser part, then the Square-Root sought (that is the Square-Root of $27 - \sqrt{704}$) will $4 - \sqrt{11}$.

Question. 3.

Two Men D and E made a Stock of *b* (165) Pounds. D's Money was in Company for *c* (12) Months, and E's Money was in for *d* (8) Months. When they shar'd Stock and Gain, D receiv'd *f* (67) Pounds, and E *g* (126) Pounds. I demand each Mau's Stock?

Before I give an Answer to the above Question, it will not be improper to lay down and Demonstrate the Principle upon which the Solution of all Questions in Fellowship with Time is founded, in the following

Lemma.

If two Men D and E make a joint Stock, but so that the Money of them both is not out for equal Times; then I say, that D's Stock Multiplyed by the Time of its being out, is to E's Stock Multiplyed by the Time of its being out, as D's Gain is to E's Gain.

Sup-

Suppose D's Stock $= a$; the Time of its being out $= c$, and his Gain $= y$.

Suppose E's Stock $= e$; the Time of its being out $= d$, and his Gain $= z$.

I say, that $ca \,..\, de :: y \,..\, z$.

Demonstration.

If D's Stock and E's Stock had been out for equal Times, then $a \,..\, y :: e \,..\, \frac{ey}{a}$ which is E's Gain, in case his Stock was out just as long as D's. Now because his Gain must be either greater or less than this, according as the Time of his Money's being out was greater or less than the Time of D's Money being out, therefore

$$c \,..\, \frac{ey}{a} :: d \,..\, z.$$

Whence arises this Equation $cz = \frac{dey}{a}$; therefore $caz = dey$.

And consequently $ca \,..\, de :: y \,..\, z$. *W. W. D.*

This being premis'd, I thus proceed to give a

Solution.

1. Suppose D's Stock $= a$; then his Stock Multiplyed by his Time is $= ca$, and his Gain is $= f - a$.
2. Then E's Stock $= b - a$, his Stock Multiplyed by his Time $= db - da$, and his Gain $= g - b + a$.

By our *Lem.*	3	$ca \,..\, db - da :: f - a \,..\, g - b + a$.
$\because$	4	$cga - cba + caa = fdb - fda - dba + daa$.
4 Transp.	5	$caa - daa + cga + fda + dba - cba = fdb$.
$5 \div \overline{c-d}$	6	$aa + \frac{cg + fd + db - cb}{c - d} a = \frac{fdb}{c-d}$.
Put	7	$h = \frac{cg + fd + db - cb}{c - d}$.
6, 7	8	$aa + ha = \frac{fdb}{c-d}$.
Comp. □.	9	$aa + ha + \frac{1}{4}hh = \frac{fdb}{c-d} + \frac{hh}{4}$.
$9 \,\omega\, 2$	10	$a + \frac{1}{2}h = \sqrt{:\frac{fdb}{c-d} + \frac{hh}{4}:}$

$10 - \frac{1}{2}b$	11	$a = -\frac{1}{2}b + \sqrt{:}\frac{fdb}{c-d} + \frac{bb}{4} : (= 55\,l.)$ $=$ D's Stock.
Then, 2,	12	$b - a =$ E's Stock $(= 110\,l.)$

Question 4.

A Merchant bought a Piece of Cloth for b (50) Shillings; and, after cutting off c (10) Yards, sold the Remainder for as much as the whole Piece cost him, by which he gain'd d ($\frac{1}{4}$) Shillings *per* Yard. The Question is, how many Yards were in that Piece, and what it cost him *per* Yard?

Suppose	1	$a =$ the Number of Yards that was in the Piece.
and	2	$e =$ the Number of Shillings that the Cloth cost him *per* Yard.
Then	3	The whole Piece cost him $ae = b$ Shillings.
and	4	$:a - c : \times : e + d : (= b) = ae = ae + ad - ce - cd$, by the Question.
$4 + ce$	5	$ce = ad - cd.$
$5 \div c$	6	$e = \frac{da}{c} - d.$
$3 \div a$	7	$e = \frac{b}{a}.$
$6 = 7$	8	$\frac{da}{c} - d = \frac{b}{a}.$
8 Reduc'd	9	$a = \frac{c}{2} + \sqrt{:}\frac{cc}{4} + \frac{bc}{d} : (= 50$ Yards).
6, 9	10	$e = -\frac{d}{2} + \sqrt{:}\frac{dd}{4} + \frac{bd}{c} : (= 1$ Shill).

Question 5.

'Tis required to find two such Numbers, that the greater of them Divided by the lesser, the Quotient may be less by b (2) than the Difference of the said two Numbers; And that their Product may exceed their Sum by c (20).

Suppose	1	$a =$ the greater Number.
and	2	$e =$ the lesser.
Then	3	$a \div e = a - e - b.$
and	4	$ae = a + e + c$, *per* Question.
$3 \times e$	5	$a = ae - ee - be.$
5,	6	$ae - a = ee + be.$
$6 \div \overline{e - 1}$	7	$a = \frac{ee + be}{e - 1}.$

$4 - a - e$	8	$ae - a - e = c$.
$7, 8$	9	$\frac{e^3 + bee - ee - be}{e - 1} (= ee + be) - e =$ $c = ee + de$ (putting d equal to $b - 1$).
9 Reduc'd	10	$\sqrt{:\frac{1}{4}dd + c:} - \frac{1}{2}d = e \; (= 4)$.

The Value of e being thus had, that of a $(= 8)$ may be found by the 7th Step.

Question 6.

Four Numbers in ÷ are required, the Sum of whose Means is $= b$ (6), and the Sum of their Extreams $= c$ (9).

Suppose	1	$a =$ second Number.
Then	2	$b - a =$ third Number.
Also	3	$b - a \,..\, a :: a \,..\, \frac{aa}{b-a} =$ first Number.
And	4	$a \,..\, b - a :: b - a \,..\, \frac{bb - 2ba + aa}{a} =$ fourth Numb.
And	5	$\frac{aa}{b-a} + \frac{bb - 2ba + aa}{a} = c$, *per* Question.
5 Reduc'd	6	$a = \frac{b}{2} \pm \sqrt{:\frac{bb}{4} - \frac{bbb}{3b+c}:} \; (= 4 \text{ or } 2)$.

The second Number $= a$ being thus found, the other three may be had by the 2d, 3d and 4th Steps: (*viz.* 2 or 4 = third, 8 or 1 = first, and 1 or 8 = fourth Number).

Question 7.

To find four Numbers in ÷ such that their common Difference may be $= b$ (2), and their Product $= c$ (945).

Put	1	$a - 1\frac{1}{2}b =$ the first Number.
Then	2	$a - \frac{1}{2}b$ will be $=$ 2d, $a + \frac{1}{2}b =$ 3d, and $a + 1\frac{1}{2}b =$ the 4th Number.
And	3	$a^4 - \frac{5}{2}bbaa + \frac{9}{16}b^4 = c$, by the Question.
$3 - \frac{9}{16}b^4$	4	$a^4 - \frac{5}{2}bbaa = c - \frac{9}{16}b^4$.
Comp. □	5	$a^4 - \frac{5}{2}bbaa + \frac{25}{16}b^4 = c - \frac{9}{16}b^4 + \frac{25}{16}b^4 = c + b^4$.
5 ω 2.	6	$a^2 - \frac{5}{4}bb = \sqrt{:c + b^4:}$
$6 + \frac{5}{4}bb$	7	$a^2 = \frac{5}{4}bb + \sqrt{:c + b^4:}$
7 ω 2.	8	$a = \sqrt{:\frac{5}{4}bb + \sqrt{c + b^4}:} \; (= \sqrt{:5 + \sqrt{945 + 16}:} = 6)$.

The Value of a being thus found, the four Numbers sought may be found by the 1st and 2d Steps: (*Viz.* $6 - 3 = 3 =$ 1st, $6 - 1 = 5 =$ 2d, $6 + 1 = 7 =$ 3d, and $6 + 3 = 9 =$ the 4th Number).

Question

Question 8.

There are five Numbers in $\div$ whose Sum is $= b$ (20), and their Product $= c$ (720). What are those Numbers?

Suppose	1	$a - 2e$ and $a - e$ to be equal to the first and second Numbers sought respectively;
then	2	a, $a + e$ and $a + 2e$ will be equal to the third, fourth, and fifth Numbers sought respectively.
Then	3	$a - 2e + a - e + a + a + e + a + 2e = b = 5a$;
and	4	$:a - 2e : \times : a + 2e : \times : a - e : \times : a + e : \times a = c = a^5 - 5a^3 ee + 4ae^4$, *per* Question.
$3 \div 5$	5	$\frac{b}{5} = a\ (= 4)$.
5, 4	6	$\frac{bbbbb}{3125} - \frac{bbb}{25} ee + \frac{4b}{5} e^4 = c$.
6×3125	7	$b^5 - 125 b^3 ee + 2500 b e^4 = 3125 c$.
7 Transp.	8	$2500 b e^4 - 125 b^3 e^2 = 3125 c - b^5$.
$8 \div 2500 b$	9	$e^4 - \frac{bb}{20} e^2 = \frac{5c}{4b} - \frac{b^4}{2500}$.
Compl. $\square$	10	$e^4 - \frac{bb}{20} e^2 + \frac{b^4}{1600} = \frac{5c}{4b} - \frac{b^4}{2500} + \frac{b^4}{1600} = \frac{5c}{4b} + \frac{9b^4}{40000}$.
10 ш 2	11	$e^2 - \frac{bb}{40} = \pm \sqrt{:\frac{5c}{4b} + \frac{9b^4}{40000}:}$
$11 + \frac{bb}{40}$	12	$e^2 = \frac{bb}{40} \pm \sqrt{:\frac{5c}{4b} + \frac{9b^4}{40000}:}$
12 ш 2	13	$e = \sqrt{:\frac{bb}{40} \pm \sqrt{\frac{5c}{4b} + \frac{9b^4}{40000}}:}$ (= $\sqrt{}$

$:10 \pm \sqrt{:45 + 36:} = \sqrt{:10 \pm \sqrt{81}:} = \sqrt{19}$ or rather $= \sqrt{1} = 1$.)

The Value of e being thus had, the five Numbers sought may be found by the 5th, 1st and 2d Steps: (*Viz.* $4 - 2\sqrt{19}$, $4 - \sqrt{19}$, 4, $4 + \sqrt{19}$, and $4 + 2\sqrt{19}$; or rather $4 - 2$, $4 - 1$, 4, $4 + 1$ and $4 + 2$; *viz.* 2, 3, 4, 5 and 6 are the five Numbers required).

Question 9.

Two Men have each a certain Number of Crowns whose Sum Subtracted from the Sum of their Squares the Remainder is =

b

b (78); But their Sum added to the Product of the two Numbers, the Sum is $= c$ (39). How many Crowns had each?

1. Suppose $a + e =$ the greater, and $a - e =$ the lesser Number of Crowns sought; then

By the Question.	2	$aa + 2ae + ee + aa - 2ae + ee - a - e - a + e = b = 2aa + 2ee - 2a.$
	3	$a + e + a - e + aa - ee = c = 2a + aa - ee.$
3 Transp.	4	$aa + 2a - c = ee.$
$4 \times \overline{2}$	5	$2aa + 4a - 2c = 2ee.$
2, 5	6	$2aa + 2aa + 4a - 2c - 2a = b = 4aa + 2a - 2c.$
$6 + 2c$	7	$2c + b = 4aa + 2a.$
$7 \div \overline{4}$	8	$\frac{2c + b}{4} = aa + \frac{1}{2}a.$
8 Reduc'd	9	$\frac{-1 + \sqrt{:8c + 4b + 1:}}{4} = a (= 6).$

The Value of a being thus known, the Value of ee, and consequently that of e will be found by the 4th Step; and then the Values of $a + e$ and $a - e$ the two Numbers songht are easily had: (Thus a being $= 6$, $aa + 2a - c = ee$ (*per* 4th Step) $= 36 + 12 - 39 = 9$; consequently $e = \sqrt{9} = 3$: Wherefore $6 + 3 = 9$ and $6 - 3 = 3$ are the two Numbers sought. *Proof*: For 12 their Sum taken from 90 the Sum of their Squares leaves 78, and added to 27 their Rectangle makes 39).

Question 10.

There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is $= b$ (16); and the Square of the Mean being added to the Product of the two Extreams is $= c$ (17). What are those Numbers?

Suppose	1	a, e, y in $\div$
Then	2	$a + y = 2e$, by the Nature of $\div$
and	3	$aa + ey = b$ } by the Question
	4	$ee + ay = c$ }
$2 - a$	5	$y = 2e - a.$
$3 - aa$	6	$ey = b - aa.$
$6 \div e$	7	$y = \frac{b - aa}{e}.$

$5=7$	8	$2e-a=\frac{b-aa}{e}$.
$8\times e$	9	$2ee-ae=b-aa$.
9 Transp.	10	$aa-ae=b-2ee$.
$4-ee$	11	$ay=c-ee$.
$11\div a$	12	$y=\frac{c-ee}{a}$.
$5=12$	13	$2e-a=\frac{c-ee}{a}$.
$13\times a$	14	$2ae-aa=c-ee$.
$10+14$	15	$ae=b+c-3ee=m-3ee$, putting $m=b+c$.
$15\div e$	16	$a=\frac{m-3ee}{e}$.
$16\odot 2$	17	$aa=\frac{mm-6mee+9e^4}{ee}$.
$17-15$	18	$aa-ae=\frac{mm-6mee+9e^4}{ee}-m+3ee$ $=\frac{12e^4-7me^2+mm}{ee}$.
$18=10$	19	$\frac{12e^4-7me^2+mm}{ee}=b-2ee$.
$19\times ee$	20	$12e^4-7me^2+m^2=bee-2e^4$.
20 Transp.	21	$14e^4-7me^2-bee=-mm$.
$21\div 14$	22	$e^4-\frac{m}{2}e^2-\frac{b}{14}ee=-\frac{mm}{14}=e^4-nee$, putting $n=\frac{1}{2}m+\frac{b}{14}$.
22d Reduc'd	23	$e=\sqrt{:\frac{n}{2}\pm\sqrt{\frac{nn}{4}-\frac{mm}{14}}}:(=\sqrt{:\frac{247}{28}}$

$\pm\sqrt{\frac{25}{784}}:=\sqrt{\frac{247\pm5}{28}}=3$ or $\sqrt{\frac{121}{14}}=\frac{121}{\sqrt{1694}}$).

e being thus found (to be $=3$ or $\frac{121}{\sqrt{1694}}$), a, by the 16th Step, will be found ($=2$ or $\frac{99}{\sqrt{1694}}$); and then y, by the 12th Step, will be also found ($=4$ or $\frac{143}{\sqrt{1694}}$).

N. B.

N.B. *The above Numbers with Negative Signs prefixt to them will likewise Solve the Question; for the first Radical Sign prefixt to the Value of* e *in the 23d Step may as well have — prefixt to it as +: So that this Question is capable of four different Answers: You may therefore conclude that the Value of* e *can't be express'd by any Equation lower than that in the 23d Step.*

Question 11.

There are three Numbers; the Square of the First of which added to the Product of the Second and Third is $=b$ (1); the Square of the Second added to the Product of the First and Third is $=c$ (2); and the Square of the Third Number added to the Product of the First and Second is $=d$ (4). What are those Numbers?

1. Suppose a, ae and aey to be equal to the First, Second and Third Numbers sought respectively.

Then	2	$aa + aaeey = b$ } by the Question.
	3	$aaee + aaey = c$ } by the Question.
	4	$aaeeyy + aae = d$ } by the Question.
$2 \div \overline{1+eey}$	5	$aa = \dfrac{b}{1+eey}$.
$3 \div \overline{ee+ey}$	6	$aa = \dfrac{c}{ee+ey}$.
$4 \div \overline{eeyy+e}$	7	$aa = \dfrac{d}{eeyy+e}$.
$5=6$	8	$\dfrac{b}{1+eey} = \dfrac{c}{ee+ey}$.
8,	9	$y = \dfrac{bee-c}{cee-be}$.
$6=7$	10	$\dfrac{c}{ee+ey} = \dfrac{d}{eeyy+e}$.
10,	11	$ceeyy - dey = dee - ce$.
$9 \odot 2$	12	$yy = \dfrac{bbe^4 - 2bce^2 + cc}{cce^4 - 2bce^3 + bbee}$.
$12 \times cee$	13	$ceeyy = \dfrac{bbce^4 - 2bcce^2 + ccc}{ccee - 2bce + bb}$.
$9 \times de$	14	$dey = \dfrac{bdee-cd}{ce-b}$.
13, 14, 11	15	$\dfrac{bbce^4 - 2bccee + c^3}{ccee - 2bce + bb} - \dfrac{bdee-cd}{ce-b} = dee - ce$.
15 Reduc'd	16	$\begin{smallmatrix}+bb\\-cd\end{smallmatrix}e^4 \begin{smallmatrix}+cc\\+bd\end{smallmatrix}e^3 - 4bce^2 \begin{smallmatrix}+bb\\+cd\end{smallmatrix}e \begin{smallmatrix}-bd\\+cc\end{smallmatrix} = 0$.

Now

Now the Numbers expressing the Values of the above given Quantities must be Substituted for them in the above Equation, and then one of the Roots (or Values of e) in that Equation must be Extracted, by the Numeral Exegesis or Converging Series, &c. And when any one of the Values of e is known, that of y will be accordingly found, by the 9th Step; and then, by the 5th, 6th, or 7th Step will likewise be found the corresponding Value of aa, and consequently of a; &c.

(Thus the Equation in the 16th Step being Numerically express'd is $-7e^4+8e^3-8ee+9e-0=0$; and one of the Values of e in this Equation is Manifestly 0: And then $y = \frac{1\times 00-2}{2\times 00-1\times 0}$ (by the 9th Step) $= \frac{-2}{-1\times 0}$ is $=\frac{2}{0}$;

wherefore $aa = \frac{2}{00+0\times\frac{2}{0}}$ (by the 6th Step) is $=1$; Consequently $a=1$ or $=-1$: And therefore 1, 0($=1\times 0$) and 2($=1\times 0\times\frac{2}{0}$); or -1, -0 and -2 are Numbers that will answer the Conditions required by the propos'd Question.

Again the above Equation Divided by $e-0$ ($=0$) quotes $-7e^3+8e^2-8e+9=0$; Consequently $7e^3-8ee+8e-9=0$ an Equation wherein one of the Values of e may, by either of Dr. *Halley*'s Theorems, be expeditiously found to be 1.1344582122, &c. then y (by the 9th Step) will be $=-.49530280+$: And then aa (by the 6th Step) $=2.758258 85+$, Consequently $a=1.6608006+$ or $=-1.6608006$, &c.

Wherefore 1.6608006 +, 1.8841089 – and –.9332044, &c. or –1.6608006, &c. –1.8841089, &c. and +.9332044 + are likewise answers to the Question propos'd.

Again $7e^3-8e^2+8e-9=0$ being Divided by $e-1.1344582122$ ($=0$) quotes $7e^2-.0587925145e+7.933302349=0$ nearly; And this Equation Divided by 7 gives $e^2-.0083989306e+1.133328907=0$ nearly, an Adfected Quadratick Equation which hath no real Root: And consequently the Question propos'd, when design'd in the above Numbers, has but four real Answers, the four Answers more being Imaginary).

☞ *See this Question otherwise solv'd in Dr.* Wallis's *Algebra*, Vol. 2 *from* Page 244 *to* Page 274.

N.B. In most Questions the best Way of Noting down the unknown Quantities is by the single ones a, e, y, &c. But in some, as in this 11th, the 9th, 8th and other Questions, it is better to design them otherwise. When therefore a Question is propos'd to be solv'd, it will be proper, before hand, to consider which is the best Method of Noting down the Quantities unknown.

Quest.

Question 12.

'Tis required to find four Numbers in ∺ whose Sum may be $= b$ (15); and the Sum of their Squares $= c$ (85).

1. Suppose the two Means to be equal to a and e.
2. Then the two Extreams will be equal to $\frac{aa}{e}$ and $\frac{ee}{a}$.

Consequently, *per* Question.

	3	$\frac{aa}{e} + a + e + \frac{ee}{a} = b.$
and	4	$\frac{a^4}{ee} + aa + ee + \frac{e^4}{aa} = c.$
$3 \times ae$	5	$a^3 + aae + aee + e^3 = bae.$
$4 \times aaee$	6	$a^6 + a^4ee + aae^4 + e^6 = caaee.$
$5 \div \overline{a+e}$	7	$aa + ee = \frac{bae}{a+e}.$
$6 \div \overline{aa+ee}$	8	$a^4 + e^4 = \frac{caaee}{aa+ee}.$
7 ⓖ 2	9	$a^4 + 2aaee + e^4 = bb \times \left.\overline{\frac{ae}{a+e}}\right\|$
$9 - 2aaee$	10	$a^4 + e^4 = bb \times \left.\overline{\frac{ae}{a+e}}\right\|^2 - 2 \times \left.\overline{ae}\right\|^2$
7, 8	11	$a^4 + e^4 = caaee \div \frac{bae}{a+e} = \frac{cae \times :a+e:}{b}$
$7 + 2ae$	12	$aa + 2ae + ee = \left.\overline{a+e}\right\|^2 = \frac{bae}{a+e} + 2ae.$
Now put	13	x for the Unknown Quantity ae,
and	14	z for the Unknown Quantity $a+e$.
10, 11, 13, 14	15	$bb \times \frac{xx}{zz} - 2xx = \frac{cxz}{b}.$
12, 13, 14	16	$zz = \frac{bx}{z} + 2x.$
$15 \times \frac{bzz}{x}$	17	$b^3x - 2bzzx = cz^3.$
$16 \times cz$	18	$cz^3 = cbx + 2czx.$
$17 = 18$	19	$b^3x - 2bzzx = cbx + 2czx.$
$19 \div x$	20	$b^3 - 2bzz = cb + 2cz.$
20 Transp.	21	$2bzz + 2cz = b^3 - cb.$
$21 \div 2b$	22	$zz + \frac{c}{b}z = \frac{1}{2}bb - \frac{1}{2}c.$

20 +

$20+\frac{cc}{4bb}$	23	$zz+\frac{c}{b}z+\frac{cc}{4bb}=\frac{bb}{2}-\frac{c}{2}+\frac{cc}{4bb}$.
21 ɯ 2	24	$z+\frac{c}{2b}=\sqrt{}:\frac{bb}{2}-\frac{c}{2}+\frac{cc}{4bb}:$
$24-\frac{c}{2b}$	25	$z=\sqrt{}:\frac{bb}{2}-\frac{c}{2}+\frac{cc}{4bb}:-\frac{c}{2b}$. ($=\frac{265}{30}-\frac{85}{30}=\frac{180}{30}=6$.)
18,	26	$x=\frac{z^3}{2z+b}$ ($=\frac{216}{12+15}=8$).

The Values of z and x being thus found, we are next to find those of a and e: Thus,

$13\div e$	27	$\frac{x}{e}=a$.
14, 27	28	$z=\frac{x}{e}+e$.
$28\times e$	29	$ze=x+ee$.
29 Transp.	30	$ee-ze=-x$.
30 Reduc'd	31	$e=\frac{1}{2}z\pm\sqrt{}:\frac{1}{4}zz-x:$ ($=3\pm1=4$ or 2).
$14-e$	32	$z-e=\frac{1}{2}z\mp\sqrt{}:\frac{1}{4}zz-x:=a$ ($=2$ or 4).
	33	The Values of the two Means a and e being thus made known, those of the Extreams, *viz.* $\frac{aa}{e}$ ($=1$ or 8) and $\frac{ee}{a}$ ($=8$ or 1) are known of Course.

N.B. *Part* XII. *begins Page* 177, *Signature* *N.

PART

PART XII.

Of several Methods of Solving high adfected Equations.

CHAP. I.

Some Preparations sometimes convenient for the Solution of high adfected Equations.

PROP. I.

To increase, or diminish the Value of the unknown Roots of an Equation by any given Quantity.

Rule.

INSTEAD of the unknown Root substitute another unknown Root less, or greater than the former by the given Quantity. Thus,

Examples.

1. To augment the Roots of this Equation $aa+ba-dd=0$ by the Quantity c.

Put $e-c=a$, then $aa+ba-dd=0$ will become $ee-2ec+cc+be-bc-dd=0$, whose Roots ($=e$) exceed them of the former ($=a$) by the Quantity c; that is $e=c+a$.

2. To Diminish the Roots of this Equation $aa+ba-dd=0$, by the Quantity c.

Put $e+c=a$, and then you'll have this Equation, $ee+2ec+cc+be+bc-dd=0$, whose Roots are less than those of the Equation propos'd by the Quantity c.

Scholium I.

By this Method of Substitution any Term, except the first, of any propos'd Equation may be destroy'd:

So, if the Equation propos'd be $x^4-bx^3+cxx-dx+f=0$;

Put $y+n=x$; then

$$\begin{array}{rrrrrcl}
y^4+4ny^3 & +6nnyy & +4n^3y & + & n^4 & = & x^4 \\
-by^3 & -3bnyy & -3bn^2y & - & bn^3 & = & -bx^3 \\
 & +cyy & +2cny & + & cnn & = & +cxx \\
 & & -dy & - & dn & = & -dx \\
 & & & + & f & = & +f
\end{array}$$

Now, 'tis plain that any Term, except the first, may be taken away from this Equation, because n was taken at pleasure; *viz.* putting $4n-b=0$, the second Term must vanish; and, putting $6nn-3bn+c=0$, the third Term will vanish. In like manner any other Term in this, or any other Equation may be destroy'd, except the first.

Therefore in taking away the second Term of any Equation, let the Index of the highest Power of the sought Root x be $=m$; and supposing the Coefficient of the second Term to be $=\mp k$; substitute $y\frac{-}{+}\frac{k}{m}$ (*viz.* $y-\frac{k}{m}$, if the said Coefficient be $+k$, but $y+\frac{k}{m}$, if it be $-k$) in the room of x every where in the propos'd Equation, and you'll have another in which y will be the Root sought, and which will want the second Term. Then, after you have found any of the Values of y, you may find accordingly that of x, for $x=y\frac{-}{+}\frac{k}{m}$.

Schol. II.

By this way of Substitution, you may add to any Equation any Term that it wants.

As,

As, supposing the Equation to be $x^4 + c^3 x - d^4 = 0$.

Put $y - c = x$; then

$$\begin{array}{r} y^4 - 4cy^3 + 6ccyy - 4c^3y + c^4 = x^4 \\ + c^3y - c^4 = + c^3x \\ - d^4 = - d^4 \\ \hline y^4 - 4cy^3 + 6ccyy - 3c^3y - d^4 = 0 \end{array}$$

PROP. II.

To Multiply or Divide the Roots of an Equation by any given Quantity; suppose q.

Rule.

Instead of the unknown Root, substitute $\frac{1}{q}$ Part, or q times another unknown Root: Thus,

Examples.

1. To Multiply the Roots of this Equation $x^3 + bx^2 - ccx = g^3$ by q.

Suppose $\frac{1}{q} \times z = x$; then the former Equation will become $\frac{z^3}{q^3} + \frac{bzz}{qq} - \frac{ccz}{q} = g^3$: And, by multiplying each Part by q^3, we have $z^3 + qbzz - qqccz = q^3g^3$, an Equation wherein z is $= qx$ in the former.

The Reason is obvious; for if $\frac{1}{q} z = x$, then $z = qx$.

2. To divide by 3 the Roots of this Equation $x^3 - 54ccx - 216d = 0$.

Suppose $3 \times z = x$, then the former Equation will become $27z^3 - 162ccz - 216d^3 = 0$: And, by dividing each Part by 27, we have $z^3 - 6ccz - 8d^3 = 0$ an Equation wherein z is $= \frac{x}{3}$ in the former.

COROLLARY.

Hence an Equation may be cleared from Fractions without increasing the Coefficient of the highest Term: Thus, if $x^3 + \frac{1}{3}x - \frac{40}{7} = 0$; that is (first reducing the Fractions $\frac{1}{3}x$ and $-\frac{40}{7}$ to a common Denominator) if $x^3 + \frac{7}{21}x - \frac{120}{21} = 0$;

I put $q = 21$, and $\frac{z}{q} = x = \frac{z}{21}$; then the former Equation will become $\frac{z^3}{9261} + \frac{7z}{441} - \frac{120}{21} = 0$; and, by multiplying each Part by 9261, we have $z^3 + 147z - 52920 = 0$.

NB. *See the Use of this* Coroll. *in the Beginning of* Chap. *5. of this Part.*

Note. The last Term but one of any Equation wanting the second Term (which, if present, may be taken away by *Prop.* 1. *Schol.* 1.) may be destroyed, without the Extraction of Roots, by supposing $\frac{q}{z}$ = the Root y of that Equation: As for Instance,

If $y^3 * - by - c = 0$; suppose $\frac{q}{z} = y$; then $z^3 + \frac{bq}{c}z^2 * - \frac{q^3}{c} = 0$.

CHAP. II.

The Solution of Cubic Equations by *Cardan's* Method.

When any Cubick Equation, having the second Term, is propos'd to be resolv'd by *Cardan's* Method, you must first destroy its second Term (by the *last Chap. Prop.* 1. *Sch.* 1.); and then you'll reduce it into one of these four following Forms or Cases, *viz.*

| 1 | a^3

1	$a^3 * + pa = q$
2	$a^3 * - pa = q$
3	$a^3 * - pa = -q$
4	$a^3 * + pa = -q$

In each of these Equations the Quantity sought a is inserted only in two different Terms, in which its Indices are treble to one another: And, in order to solve all Equations thus qualified at once, let us suppose the reduc'd Equation to be represented by $x^n + bx^{\frac{n}{3}} = c$, in which the Values of b and c may be equal to any given Quantities whatsoever, Affirmative or Negative; and that of n is indetermin'd, but is in all original Cubick Equations $= 3$, and x^n is understood to have the Sign $+$ prefix'd to it.

Solution.

If	1	$x^n + bx^{\frac{n}{3}} = c$
Suppose	2	$z = x^{\frac{n}{3}}$
1, 2	3	$z^3 + bz = c$
Suppose	4	$e + y = z$
3, 4	5	$e^3 + 3eey + 3eyy + y^3 + b \times : e + y : = c$
Now suppose	6	$-3ey = b$
5, 6	7	$e^3 + 3eey + 3eyy + y^3 - 3eey - 3eyy = c = e^3 + y^3$
6 ÷ − 3e	8	$y = -\frac{b}{3e}$
8 ⊙ 3	9	$yyy = -\frac{bbb}{27e^3}$
7, 9	10	$c = e^3 - \frac{bbb}{27eee}$
10 × e^3	11	$ce^3 = e^6 - \frac{1}{27}b^3$
11 Transp.	12	$\frac{1}{27}b^3 = e^6 - ce^3$
Compl. □.	13	$\frac{1}{4}cc + \frac{1}{27}b^3 = e^6 - ce^3 + \frac{1}{4}cc$

13 w 2

13 w 2.	14	$\sqrt{:\frac{1}{4}cc+\frac{1}{27}b^3:}=e^3-\frac{1}{2}c$
$14+\frac{c}{2}$	15	$\frac{1}{2}c+\sqrt{:\frac{1}{4}cc+\frac{1}{27}b^3:}=e^3$
15 w 3	16	$\sqrt[3]{:\frac{1}{2}c+\sqrt{\frac{1}{4}cc+\frac{1}{27}b^3}:}=e$
8, 16	17	$y=-\dfrac{b}{3\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}:}}$
16, 17, 4	18	$\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}-\dfrac{\frac{1}{3}b}{\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}:}}=z=x^{\frac{n}{3}}$
18 w $\frac{n}{3}$.	19	$\sqrt[\frac{n}{3}]{:\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}-\dfrac{\frac{1}{3}b}{\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}:}=x$

Scholium.

As the Cube Root of any Quantity k is not only $\sqrt[3]{k}$ or $1\times\sqrt[3]{k}$, but also $:-\frac{1}{2}+\sqrt{-\frac{3}{4}}:\times\sqrt[3]{k}$, and $:-\frac{1}{2}-\sqrt{-\frac{3}{4}}:\times\sqrt[3]{k}$; so the Value of z in the Equation $z^3+bz=c$ is not only $1\times\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}-\dfrac{\frac{1}{3}b}{1\times\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}:}}$ (as in the foregoing 18th Step); but also $\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}-\dfrac{\frac{1}{3}b}{\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}:}}$, And

$$\overline{-\tfrac12-\sqrt{-\tfrac34}\times\sqrt[3]{:\tfrac{c}{2}+\sqrt{\tfrac{cc}{4}+\tfrac{b^3}{27}}:}-}$$

$$\frac{\tfrac13 b}{-\tfrac12-\sqrt{-\tfrac34}\times\sqrt[3]{:\tfrac{c}{2}+\sqrt{\tfrac{cc}{4}+\tfrac{b^3}{27}}:}}$$: For all three are but the ſame Values of z differently expreſs'd; the firſt of them being ſimpleſt, is therefore the beſt.

That theſe are the three Values of z in the Equation $z^3+bz=c$ I demonſtrate; thus:

Suppoſe $u=-\frac12+\sqrt{-\frac34}$, $v=-\frac12-\sqrt{-\frac34}$, and $w=\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}$; then the foregoing Values of z will become $w-\frac{\frac13 b}{w}$, $uw-\frac{\frac13 b}{uw}$, and $vw-\frac{\frac13 b}{vw}$ reſpectively; wherefore $z-w+\frac{\frac13 b}{w}=0$, $z-uw+\frac{\frac13 b}{uw}=0$, and $z-vw+\frac{\frac13 b}{vw}=0$: And the Products of theſe two laſt Equations will be found, when reduc'd, to be

$$z^2\begin{array}{l}+w\\-\frac{\frac13 b}{w}\end{array}z\begin{array}{l}+ww\\+\frac13 b\\+\frac{\frac19 bb}{ww}\end{array}=0$$

($u+v$ being $=-1$, $vu=1$, and $u^2+v^2=-1$); and theſe Products multiplied by $z-w+\frac{\frac13 b}{w}=0$ give $z^3+bz-w^3+\frac{\frac{1}{27}b^3}{w^3}=0$; that is $z^3+bz-c=0$; for $-w^3+\frac{\frac{1}{27}b^3}{w^3}$ may be eaſily prov'd to be $=-c$. *Q. E. D.* ☞ *See Part* IX.

Corollary.

From the third and eighteenth foregoing Steps you may deduce particular Theorems (but not the ſame with *Cardan's*) for ſolving each of the four preceeding Caſes:

So; 1. If $a^3+pa=q$; then (b in the third Step being $=p$ in this Equation, and c in the former $=q$ in the latter

latter, by the 18th Step) $a = \sqrt[3]{:\frac{q}{2} + \sqrt{\frac{qq}{4} + \frac{p^3}{27}}:} - \frac{\frac{1}{3}p}{\sqrt[3]{:\frac{q}{2} + \sqrt{\frac{qq}{4} + \frac{p^3}{27}}:}}$

Also; 2. If $a^3 - pa = q$; then ($-p$ being $= b$, and $q = c$)

$$a = \sqrt[3]{:\frac{q}{2} + \sqrt{\frac{qq}{4} - \frac{p^3}{27}}:} + \frac{\frac{1}{3}p}{\sqrt[3]{:\frac{q}{2} + \sqrt{\frac{qq}{4} - \frac{p^3}{27}}:}}$$

Again; 3. If $a^3 - pa = -q$; then

$$a = \sqrt[3]{:-\frac{q}{2} + \sqrt{\frac{qq}{4} - \frac{p^3}{27}}:} + \frac{\frac{1}{3}p}{\sqrt[3]{:-\frac{q}{2} + \sqrt{\frac{qq}{4} - \frac{p^3}{27}}:}}$$

Ex. 1. *Case* 1.

If $i^3 + 3ii + 9i = 13$, and one of the Values of i be required.

First, to destroy the second Term, I suppose $a - 1 = i$, then the foregoing Equation will become $a^3 + 6a = 20$: And, by the first particular Theorem, $a = \sqrt[3]{:10 + \sqrt{100 + 8}:} - \frac{2}{\sqrt[3]{:10 + \sqrt{100 + 8}:}} = \sqrt[3]{20.392304845413} + - \frac{2}{\sqrt[3]{20.392304845413} +} = 2.73205$ &c. $- \frac{2}{2.73205 \text{ &c.}} = 2.73205$ &c. $- .73205$ &c. $= 2$: But i is (by supposition) $= a - 1$; consequently i is $= 1$. The two other Values of i are imaginary.

Ex. 2. *Case* 2.

If $a^3 - 21a = -20$; then $a = \sqrt[3]{:-10 + \sqrt{100 - 343}:} + \frac{7}{\sqrt[3]{:-10 + \sqrt{-243}:}} = 2 + \sqrt{-3} + \frac{7}{2 + \sqrt{-3}} = 2 + \sqrt{-3} + 2 - \sqrt{-3} = 4$; Or $a =: -\frac{1}{2} + \sqrt{-\frac{3}{4}} : \times : 2 + \sqrt{-3} : + \frac{7}{: -\frac{1}{2} + \sqrt{-\frac{3}{4}} : \times : 2 + \sqrt{-3} :} = -2\frac{1}{2} + \frac{1}{2}\sqrt{-3} - 2\frac{1}{2} - \frac{1}{2}\sqrt{-3} = -5$: or a is $= : -\frac{1}{2} -$

$$\tfrac{1}{2} - \sqrt{-\tfrac{3}{4}} :\times: 2 + \sqrt{-3} :+ \frac{7}{:-\tfrac{1}{2} - \sqrt{-\tfrac{3}{4}} :\times: 2 + \sqrt{-3}:}$$
$$= \tfrac{1}{2} - \tfrac{3}{2}\sqrt{-3} + \tfrac{1}{2} + \tfrac{3}{2}\sqrt{-3} = 1.$$

Tho' the foregoing and *Cardan*'s Theorems are equally true, yet, as to the Practick Part, inaſmuch as Diviſion is eaſier than the Extraction of the Cube-Root, the former are preferable, except only when the Value of $\frac{cc}{4} + \frac{b^3}{27}$ is $\sqsupset 0$: In that reſpect they were, by ſome, deem'd impracticable and falſe; but you may ſee by the laſt Example that thoſe are not ſo: 'Tis true the Method of applying them to Practice, in Caſes of this Nature, is not perfect; it being only by Trials I came to diſcover that $2 + \sqrt{-3}$ is the Cube-Root of $-10 + \sqrt{-243}$.

But ſince the Extraction of the Cube-Root of $-10 + \sqrt{-243}$ is more difficult than finding the Values of a in the Equation $a^3 - 21a = -20$, I will now ſhew how by knowing the latter you may perform the former. Thus

Suppoſe the Cube-Root of $-10 + \sqrt{-243}$ to be $= \tfrac{1}{2}a + \sqrt{-y}$ (that the firſt Member is always $= \tfrac{1}{2}a$ is apparent from * *Cardan*'s Theorem); then $\tfrac{1}{8}a^3 + \tfrac{3}{4}aa\sqrt{-y} + \tfrac{1}{2}a \times -y\,(-\tfrac{3}{2}ay) - y\sqrt{-y} = -10 + \sqrt{-243}$: And the rational Part of the firſt Part of this Equation muſt be equal to the Rational Part of the latter; *viz.* $\tfrac{1}{8}a^3 - \tfrac{3}{2}ay$ is $= -10$; that is (knowing one of the Values of a to be 4) $8 - 6y = -10$; therefore $y = 3$: And conſequently $\tfrac{1}{2}a + \sqrt{-y} = 2 + \sqrt{-3}$ is the Cube-Root of $-10 + \sqrt{-243}$.

Or, ſince one of the Values of a is likewiſe 1; therefore the Cube-Root of $-10 + \sqrt{-243}$ will be alſo found (by this Method) to be $= \tfrac{1}{2} + \sqrt{-\tfrac{27}{4}}$ (for $-\tfrac{3}{2}ay$ is $= \tfrac{1}{2}a \times \sqrt{-y} \times \sqrt{-y}$) $= \tfrac{1}{2} - \sqrt{-\tfrac{27}{4}}$.

Or laſtly, ſince one of the Values of a is alſo -5; the Cube-Root of $-10 + \sqrt{-243}$ will, in like manner, be found $= -2\tfrac{1}{2} + \sqrt{-\tfrac{3}{4}}$.

I will now proceed to the Diſcovery of that general Theorem from which *Cardan*'s are deducible; and then ſhew how Equations, in which the Value of $\tfrac{1}{4}cc + \tfrac{1}{27}b^3$ is negative, may be thereby ſolv'd.

Having proceeded as far as the 16th Step incluſive of the foregoing Operation, you'll from the 7th and 15th find for the

* *See the following Page.*

	17	$\frac{c}{2}+\sqrt{:\frac{cc}{4}+\frac{b^3}{27}:}+y^3=c.$
17 Reduc'd	18	$y=\sqrt[3]{:\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}$
16, 18, 4	19	$\sqrt[3]{:\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}+\sqrt[3]{:\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}:}=z=x^{\frac{n}{3}}$
19 ɯ $\frac{n}{3}$	20	$\sqrt[\frac{n}{3}]{:\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}+\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}:}=x.$

Scholium.

From the 3d and 19th Steps *Cardan*'s Theorems may be easily deduc'd.

Example in which the Value of $\frac{1}{4}cc+\frac{1}{27}b^3$ is $\sqsupset 0$.

If $a^3-12.03a=16$;
that is, if $a^3-pa=q$, and that $\frac{1}{4}qq-\frac{1}{27}p^3$ is $\sqsupset 0$;
'Tis required to find one of the Values of a proxime by the Help of *Cardan*'s Method.

The particular Theorem for evolving this Equation deducible from the 19th Step, is

$$a=\sqrt[3]{:\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{p^3}{27}}:}+\sqrt[3]{:\frac{q}{2}-\sqrt{\frac{qq}{4}-\frac{p^3}{27}}:};$$

by the Help of which one of the Values of a may be found near the Truth, thus:

Suppose $\frac{q}{2}=r$, and $\sqrt{:\frac{qq}{4}-\frac{p^3}{27}:}=\sqrt{-s}$; then the Values of a in the said Theorem will become $=\sqrt[3]{:r+\sqrt{-s}:}+\sqrt[3]{:r-\sqrt{-s}:}$ And (by *Part* XV. *Chap.* 1.) $\sqrt[3]{:r\pm\sqrt{-s}:}=r^{\frac{1}{3}}+\frac{1}{3}\times r^{\frac{1}{3}-1}\times\pm\sqrt{-s}+\frac{\frac{1}{3}}{1}\times\frac{\frac{1}{3}-1}{2}\times r^{\frac{1}{3}-2}\times -s+\frac{\frac{1}{3}}{1}\times\frac{\frac{1}{3}-1}{2}\times\frac{\frac{1}{3}-2}{3}\times r^{\frac{1}{3}-3}\times\pm\sqrt{-s}^{\,3}+\frac{\frac{1}{3}}{1}\times\frac{\frac{1}{3}-1}{2}\times\frac{\frac{1}{3}-2}{3}\times\frac{\frac{1}{3}-3}{4}\times r^{\frac{1}{3}-4}\times +ss$, &c.

therefore

therefore $a\ (=\sqrt[3]{:r+\sqrt{-s}:}+\sqrt[3]{:r-\sqrt{-s}:})=2\times r^{\frac{1}{3}}*+2\times\frac{1}{3}\times\frac{\frac{1}{3}-1}{2}\times r^{\frac{1}{3}-2}\times -s*+2\times\frac{1}{3}\times\frac{\frac{1}{3}-1}{2}\times\frac{\frac{1}{3}-2}{3}\times\frac{\frac{1}{3}-3}{4}\times r^{\frac{1}{3}-4}\times ss$, &c. $=2\times r^{\frac{1}{3}}+\frac{2s}{9\times r^{\frac{5}{3}}}-\frac{20ss}{243\times r^{\frac{11}{3}}}$, &c. *sine fine*: Consequently one of the Values of a in the propos'd Equation is (because r is equal to 8, and $\sqrt{-s}$ equal to $\sqrt{-.481201}$) $=2\times 8^{\frac{1}{3}}\ (=2\times 2)+\frac{.962402}{9\times 2^{5}}-\frac{4.631089}{243\times 2^{11}}$, &c. $=4+\frac{.962402}{288}-\frac{4.631089}{497664}$, &c. $=4+.003341$ &c. $-.000009$ &c, &c. $=4.003332$, &c.

Having thus long dwell'd upon this Subject, and render'd it tedious to the Learner, and Press, I am sorry I must leave it imperfect as it is: For could the Canon in Page 182 be as effectually applied to Cases wherein $\frac{1}{27}b^3+\frac{1}{4}cc$ is $\sqsupset 0$ (in which Cases all the Roots in all original Cubicks are always real) as it could to the Cases wherein $\frac{1}{27}b^3+\frac{1}{4}cc$ is $\sqsubset 0$, we should have a perfect Method for extracting the Roots of all sorts of Cubick Equations; and consequently of all Biquadraticks, by the following Chap. But the above Method of Series (tho' it be the best I could deduce from the foregoing Theorems for to apply to the Cases herein first mention'd) is, in most Examples, very prolix and troublesom: And therefore I must refer such Equations to be evolved by the numeral Exegesis, the converging Series, or by the * Trisection of an Angle.

* *See Prob.* 9. *chap.* 3. *Part* II. *Book* II.

CHAP. III.

A Rule of *des Cartes*'s for dissolving a Biquadratick into two Quadraticks.

ANY Biquadratick Equation, which hath the second Term, being propos'd to be dissolv'd into two Quadraticks, must have the second Term first destroy'd (by *Chap.* 1

Prop. 1. *Sch.* 1.); and let the Equation thence produc'd be suppos'd equal to $x^4 * + qxx + rx + s = 0$; in which Equation x^4 is suppos'd to have the Sign $+$ prefix'd to it; But the Values of q, r and s may either of them be affirmative or negative. Let us suppose this Equation to be produc'd by the two Quadraticks $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$; in which Equations x^2 is understood to have the Sign $+$ prefix'd to it: And, to the End that the second Term should be wanting in their Product, ex must have the Sign $+$ in one Equation, and $-$ in the other, and the Values of f and g are to be determin'd in the following manner: Then $x^4 + qx^2 + rx + s = x^4 * \begin{matrix}+f\\-ee\\+g\end{matrix} x^2 \begin{matrix}-ef\\+eg\end{matrix} x + fg$: And, by equating their respective Terms, we have $q = f + g - ee$, $r =: g - f : \times e$, and $s = fg$; wherefore $q + ee = f + g$, $\frac{r}{e} = g - f$; consequently $\frac{q + e^2 + \frac{r}{e}}{2} = g$, and $\frac{q + e^2 - \frac{r}{e}}{2} = f$, and the Products of the two last Steps are $\frac{qq + 2qe^2 + e^4 - \frac{rr}{ee}}{4} = fg = s$; which Equation, when reduc'd, gives $e^6 + 2qe^4 \begin{matrix}+qq\\-4s\end{matrix} e^2 = rr$. Now suppose $e^2 = y$, and then you'll reduce the foregoing Equation to this Cubick one $y^3 + 2qy^2 \begin{matrix}+qq\\-4s\end{matrix} y = rr$, whose second Term may be destroy'd, and the Root of the Equation thus had extracted, by the foregoing *Chap.* and consequently one of the Values of y found; as also of $e = \sqrt{y}$; as also of $f = \frac{q + e^2 - \frac{r}{e}}{2}$; and lastly of $g = \frac{q + ee + \frac{r}{e}}{2}$: And, by extracting the Roots of the two Quadraticks, $x^2 + ex + f = 0$ and $x^2 - ex + g = 0$, you will have the four Roots of the Biquadratick $x^4 + qx^2 + rx + s = 0$; to wit, $x = -\frac{1}{2}e \pm \sqrt{: \frac{1}{4}ee - f :}$, and $x = \frac{1}{2}e \pm \sqrt{: \frac{1}{4}ee - g :}$

Example.

Example.

If $a^4 - 2a^3 - a + 2 = 0$, and the Values of a be required.

First, in order to destroy the second Term, suppose $a = x + \frac{1}{2}$; then the foregoing Equation will become $x^4 * - \frac{3}{2}x^2 - 2x + \frac{21}{16} = 0$, in which $-\frac{3}{2}$ is $= q$, $-2 = r$, and $\frac{21}{16} = s$; wherefore $y^3 + 2qy^2 \begin{array}{l}+qq\\-4s\end{array} y - rr\ (=0) = y^3 - 3y^2 - 3y - 4 = 0$: And likewise to destroy the second Term of this Equation, suppose $y = v + 1$, then it will become $v^3 - 6v - 9 = 0$: Whence v will be found, by the last *Chap.* $= \sqrt[3]{} : 4\frac{1}{2} + \sqrt{20\frac{1}{4} - 8} : + \sqrt[3]{} : 4\frac{1}{2} - \sqrt{12\frac{1}{4}} : = \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = 3$: And $v + 1 = y = 3 + 1 = 4$. Now y being thus found, its Square-Root $= 2$ is $= e$, and $\dfrac{q + ee - \frac{r}{e}}{2}$ that is, $\dfrac{-\frac{3}{2} + 4 + 1}{2}$, or $1\frac{3}{4} = f$, and $\dfrac{q + ee + \frac{r}{e}}{2}$, or $\frac{3}{4} = g$: Wherefore the Equations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$, are equal to $x^2 + 2x + 1\frac{3}{4} = 0$, and $x^2 - 2x + \frac{3}{4} = 0$ respectively: Whence $x = -1 \pm \sqrt{-\frac{3}{4}}$, and $x = 1 \pm \sqrt{\frac{1}{4}} = \left\{\begin{array}{l}1\frac{1}{2}\\ \frac{1}{2}\end{array}\right\}$: But a is, by Supposition, $= x + \frac{1}{2}$; consequently the four Values of a in the propos'd Equation are $-\frac{1}{2} + \sqrt{-\frac{3}{4}}$, $-\frac{1}{2} - \sqrt{-\frac{3}{4}}$, 2 and 1.

M. Tschirnhaus, *in* Act. Erud. Lips. *publish'd a Method of solving Equations by destroying all the intermediate Terms, which I will not insert here, because it is very tedious; and, as to what relates to Cubicks, less practicable, in every Case, than* Cardan's.

CHAP.

CHAP. IV.

The Solution of Equations by *Stevinus*'s Method.

THE Roots of Equations are, by this Method, found out by (sometimes frequent) Trials: Thus,

Example I.

Suppose the propos'd Equation to be $a^3 - 9aa + 26a = 24$; 'tis required to find the Values of a therein.

First, I suppose $a = 1$; and, working according to the Equation, find that $a^3\,(1) - 9\,aa\,(-9) + 26a\,(+26) = 18$; but it ought to be $= 24$; wherefore I conclude a is $\sqsubset 1$.

I try again, and suppose $a = 2$; then will $a^3\,(+8) - 9\,aa\,(-36) + 26\,a\,(+52) = 24$, which answers my Desire, and gives me one real Value of a: After which I may divide the Equation $a^3 - 9\,aa + 26\,a - 24 = 0$ by $a - 2$, which will bring it down to a Quadratick; or I may proceed further in the same Method, and find also that 3 and 4 are the two other real Roots.

Example II.

If this irregular Equation was propos'd (where also the absolute Number is a Fraction) $x^4 + 5x = 184638.6801$.

I can discover at first sight almost, that a must at least $= 10$, and trying with 10, I find it too little; but, trying with 100, I find that by much too great: Proceeding again, I find 30 too much; I try with 20, and find it by something too small; but 21 I find too big: Wherefore I know that x must be $= 20$ with some Fraction annex'd; and at last I discover 20.7 to be the very Root sought; or, at least, one of the Roots of the propos'd Equation.

LEMMA to CHAP. V.

How to find all the Divisors of a propos'd Number or Quantity.

RULE. Divide the propos'd Quantity or Number by the least of its Divisors that is $\sqsubset 1$, and the Quotient by the least

least of its Divisors that is $\sqsubset 1$, and so on 'till you have 1 for a Quotient, and you'll have all the prime Divisors of the propos'd Number or Quantity: Then multiply each two, three, four, &c. of the prime Divisors into themselves continually, and the several Products are the compound Divisors.

Examples.

I. So, if all the Divisors of 60 were required. First, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and the Quotient 5 by 5: Then the prime Divisors are 1, 2, 2, 3, 5; and the compound ones, produc'd by each two (always omitting Unity) are 4, 6, 10, 15; by each three are 12, 20, 30; by each four (or by all) 60.

II. Again, if all the Divisors of $21abb$ were required. Divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and the Quotient b by b: Then the prime Divisors are 1, 3, 7, a, b, b; and the compound ones, produc'd by each two are 21, $3a$, $3b$, $7a$, $7b$, ab, bb; by each three $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; by each four $21ab$, $21bb$, $3abb$, $7abb$, and by all five $21abb$.

III. In like manner all the Divisors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$; $2a$, $2bb - 6ac$, $abb - 3aac$; $2abb - 6aac$,

CHAP. V.

John Kersey's Method of finding the Roots of some Affected, Cubick, Biquadratick, &c. Equations.

FIRST prepare the propos'd Equation thus; *viz.*

1. If the Coefficient of the highest Power of the unknown Root be greater than 1, divide the whole Equation by that Coefficient; and then

2. If any of the Terms be Fractions, multiply the unknown Root by such a Number as will give an Integer Product, and the Coefficient of its highest Power $= 1$ (by *Propos. II. Chap.* 1.) And the Root of the Equation (whether it

it be at first, or by these Directions thus prepar'd, call a. Then

3, Reduce all the Terms of this Equation to one Side of it, and the other Side will be o.

Then find all the Divisors of the absolute Number in the Equation so reduc'd, and try whether any of those Divisors connected to the unknown Root a, by — or + will divide the total Sum of the reduc'd Equation without leaving a Remainder: For when such Division succeeds, either the known Part of the Residual, or Binominal Divisor, with a contrary Sign, is the desired Value of the Root a, or, at least, the Quotients give an Equation whose first Term hath fewer Dimensions by 1, than the Equation divided: And if this Equation contains three or more Dimensions, let it be examin'd by Division, as before; and so on. By which Divisions the Roots of the propos'd Equation may be sometimes made known, or the Equation may be reduc'd to a Quadratick one; and then the sought Root will be found by the Canons given for solving Adfected Quadraticks.

Examples.

I. If $a^3-15aa+74a-120=0$; what are the Values of a?

The Divisors of 120, the absolute Number, are 1, 2, 4, 8, 3, 6, &c. Then I try whether $a-1$, $a+1$, $a-2$, $a+2$, $a-3$, or $a+3$ will divide the Equation without leaving a Remainder: But, finding that neither of them will do, I try next with $a-4$, which will exactly do; and therefore 4 is one affirmative Value of the Root a: And the Quotients being $aa-11a+30=0$, the other two Roots will be found to be 5 and 6, by dividing $aa-11a+30=0$ by $a-5$, or by $a-6$; or by the Canon given for solving the third Case of Adfected Quadratick Equations.

II. Let it be required to find the Roots of this Equation $a^3-36aa+2a-72=0$.

The absolute Number 72 can be exactly divided by 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72; wherefore the propos'd Equation is to be divided by a — and + 1, a — and + 2, &c. to find such of them as will exactly do it: But, since here are a great many Divisors, and that (by the Composition of Equations) there can be, at most, but three such Divisors, which will exactly divide the propos'd Equation; you

you may try with a great many of those Divisors before you find any of the three sought: * Wherefore, to save your self a great deal of this Trouble, transform the propos'd Equation into another, each of whose Roots shall be more or less than those of the propos'd one by a given Number, 1 is generally the most convenient. Suppose therefore $a = x - 1$, then the above Equation will become $x^3 - 39xx + 77x - 111 = 0$, an Equation whose Roots are each by 1 more than those of the propos'd one: And the Divisors of its last Term are 1, 3, 37, and 111. But, since $a = x - 1$, it is evident that if any of the Values of x be 1, 3, 37, 111, or -1, -3, -37, -111, (as it, or they, must be, if it has any rational one) that, or those of a will be 0, 2, 36, 110, or -2, -4, -38, -112: But, by what was before said, all the Rational Values of a are inserted among the following ones, *viz.* 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72, or -1, -2, -3, -4, &c. Consequently, if a has any Rational Value in the propos'd Equation, it, or they must be 2, 36, or -2, -4: Wherefore you need now try to divide the Equation propos'd only by $a - 2$, $a - 36$, $a + 2$, and $a + 4$: Wherefore I try first to divide it by $a - 2$; but that not succeeding, I try next to divide it by $a - 36$, which exactly does, the Quotient being $a^2 + 2 = 0$, an Equation wherein a is $= \sqrt{-2}$, and $-\sqrt{-2}$; and consequently the three Roots or Values of a in the propos'd Equation are 36, $\sqrt{-2}$, and $-\sqrt{-2}$.

* Mr. *Waessenar*'s Method.

CHAP. VI.

The Solution of Affected Equations by Sir *Isaac Newton*'s Method.

THERE is an universal Method of extracting Roots, either in Numbers or Symbols, invented by Sir *Isaac Newton*, which you may find in Pages 381, 382, and 383 of Dr. *Wallis*'s Algebra; which is to this effect.

First, Find the first or greatest Member of the Root sought y; and, if it not be $= y$, let that Member $+ p$ be suppos'd $= y$; then, having substituted this Binomial and its respective Powers for y, and its Powers in the Equation, collect its several Terms into one Sum: Then find the second Mem-

ber of that Root, which is done (in some Cases, but not in all in the Beginning of the Operation) by dividing the first Term (or absolute Number, or known Quantity) of the said Sum by the Coefficient of p in the second Term thereof, and let the Quotient affected with the contrary Sign to what it has $+q$ be suppos'd $=p$: Then proceed with this Binomial or Residual in respect of p, as you did with the other in respect of y: And so on.

Note. When you have found three, four, or more of the first Figures, or Members of the Root, you may find as many, or almost as many more by dividing the first Term of the last Sum by the Coefficient of the second Term.

Example I.

If $y^3 - 2y - 5 = 0$; 'tis required to find one of the Values of y nearly.

$y^3 - 2y - 5 = 0\,(2.1 - .00544852 = 2.09455148 = y$ nearly.	
$2 + p = y$ ∴	$y^3 = 8 + 12p + 6pp + p^3$ $-2y = -4 - 2p$ $-5 = -5$
Sum	$-1 + 10p + 6pp + p^3$
$.1 + q = p$ ∴	$p^3 = .001 + .03q + .3qq + q^3$ $6pp = .06 + 1.2q + 6.qq$ $10p = 1. + 10q$ $-1 = -1$
Sum	$.061 + 11.23q + 6.3qq + q^3$
$-.0054 + r = q$ ∴	$q^3 = -.0000001 + .000$ &c. $6.3q^2 = .0001837 - .068r$, &c. $11.23q = -.060642 + 11.23r$ $.061 = .061$
Sum	$.0005416 + 11.162r$. &c.
$-.00004852 + s = r$	

Example II.

If $y^3 + axy + aay - x^3 - 2a^3 = 0$. *Quære* y *proxime.*

$y^3 +$

$$y^3 + axy + aay - x^3 - 2a^3 = 0 \left(a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512aa}, \text{ \&c. } \textit{sine fine} = y. \right.$$

$a + p = y \therefore$

$$\begin{aligned} y^3 &= a^3 + 3aap + 3app + p^3 \\ axy &= aax + axp \\ aay &= a^3 + aap \\ -x^3 &= -x^3 \\ -2a^3 &= -2a^3 \end{aligned}$$

$$\text{Sum} \quad \begin{matrix} aax + 4aa \\ -x^3 + ax \end{matrix}\; p + 3app + p^3$$

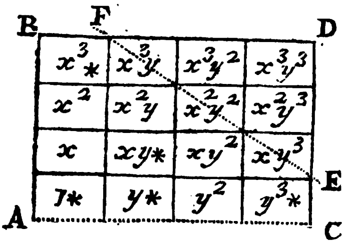
$-\frac{x}{4} + q = p \therefore$

$$\begin{aligned} p^3 &= -\tfrac{1}{64}x^3 + \tfrac{3}{16}xx, \text{ \&c.} \\ 3app &= \tfrac{3}{16}ax^2 - \tfrac{3}{2}ax \\ axp &= -\tfrac{1}{4}ax^2 + ax\,q \\ 4aap &= -aax + 4aa \\ aax &= aax \\ -x^3 &= -x^3 \end{aligned}$$

$$\text{Sum} \quad \begin{matrix} -\tfrac{1}{16}ax^2 \\ -\tfrac{65}{64}x^3 \end{matrix} \begin{matrix} +4aa \\ -\tfrac{1}{2}ax \\ +\tfrac{3}{16}xx \end{matrix}\; q, \text{ \&c.}$$

$$\frac{xx}{64a} + \frac{131x^3}{512aa} + r = q$$

The greatest Difficulty in this *Ex.* 2. is to find the first Member of the Root, which may be done; thus



Let a Parallelogram be (or suppos'd to be) describ'd; and let it be divided into as many similar small Parallelograms as are requisite: Then denominate each of these * Parallelograms from the Dimensions of the two undetermined Quantities of the Equation, as x and y (of which y denotes the Root to be extracted, and x the other undetermin'd Quantity) increasing regularly (as in the annex'd Figure) from the ▭ 1. Then mark each ▭ which answers the Terms of the propos'd Equation with an Asterisk. Then apply a Rule to the respective Corners or Angles of any two of the exterior ▭s thus mark'd

mark'd, and with the Terms of the Equation anſwering them of the two, or more, □s thus touch'd by the Rule, make a ſuppos'd Equation.

Thus a Rule laid on the Corner C and A of the mark'd exterior □s y^3 and 1 touches likewiſe the reſpective Corner of the mark'd □ y; whence the propos'd Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$ exhibits $y^3 + aay - 2a^3 = 0$; and therefore you have $y = a$ for the firſt Member of the Root to be extracted.

And a Rule laid on the Corners E and F of the mark'd exterior □s y^3 and x^3, and touching the reſpective Corner of no other mark'd □, gives $y^3 - x^3 = 0$; conſequently $y = x$, &c.

Theſe are the two different Methods of extracting the Root y, the former of which is preferable when a is ⊏ x; but the latter is beſt when x is ⊏ a. As to the remaining Manner of applying the Rule as is above directed, *viz.* to the Corners B and A of the mark'd □s x^3 and 1, the Reſult of ſuch a Poſition of the Rule is to find the Root x, not y.

PART

PART XIII.

How to raiſe Canons for finding the Sums of the Powers of an Arithmetical Progreſſion continued.

THEOREM I.

IN a Series of Units (as 1, 1, 1, 1, &c.) if the Number of Terms $= n$ be multiplied by either of them, the Product $= n$ will be equal to the Sum of all the Terms in the Series.

This is evident from the Nature of Multiplication.

THEOREM II.

In a Series of Numbers in Arithmetical Progreſſion increaſing, whoſe firſt Term is = to its common Exceſs = 1, and Number of Terms $= n$ (as 1, 2, 3, 4, &c. and n) ; if to the laſt Term $= n$ you add the firſt 1, and multiply the Sum by half the Number of Terms, the Product $= \frac{nn+n}{2}$ is equal to the Sum of the Series.

This has been demonſtrated in Arithmetical Progreſſion.

THEOREM III.

In a Series of Squares whoſe Sides or Roots are in an Arithmetical Progreſſion increaſing, whoſe firſt Term and common Exceſs are each = 1, and Number of Terms $= n$, (as 1^2, 2^2, 3^2, 4^2, &c. and n^2) I ſay that $\frac{n^3 + \frac{3}{2}nn + \frac{1}{2}n}{3}$ is = to their Sum $z = 1 + 4 + 9 + 16 +$ &c. $+ nn$.

DE-

DEMONSTRATION.

It is manifest that z is $=$ to the Difference of the Sums of the two next following Series; each of whose Ranks is the Sum of an $\div$, *viz.*

$$=\left\{\begin{array}{l}\textit{Greater Series.}\\ 1+2+3+4+\&c.+n\\ 1+2+3+4+\&c.+n\\ 1+2+3+4+\&c.+n\\ 1+2+3+4+\&c.+n\\ \overbrace{\qquad\text{\&c. to } n \text{ Ranks continued.}\qquad}\end{array}\right\}-\left\{\begin{array}{l}\textit{Lesser Series.}\\ 0\\ 1\\ 1+2\\ 1+2+3\\ \overbrace{\text{\&c. to } n \text{ Ranks continued.}}\end{array}\right\}$$

Difference of the greater and lesser Series.

$$=\left\{\begin{array}{r}1+2+3+4+\&c.+n\\ 2+3+4+\&c.+n\\ 3+4+\&c.+n\\ 4+\&c.+n\\ \overbrace{\qquad\text{\&c. to } n \text{ Ranks continued.}\qquad}\end{array}\right\}$$

For $1=1^2$, $2+2=2^2$, $3+3+3=3^2$, $4+4+4+4=4^2$, &c.

And $1+2+3+4+\&c.+n$ (or the Sum of 1, 2, 3, 4, &c. to n Terms continued) is (by *Theorem* the 2d) $=\frac{nn+n}{2}$; wherefore the Sum of the greater Series (being n times: $1+2+3+4+\&c.+n$:) is $=\frac{n^3+nn}{2}$.

In the next Place we are to find the Sum of the lesser Series; in order to which consider, its first Rank being o, its

2d

2^d Rank 1, its 3^d Rank $1+2$, its 4^{th} Rank $1+2+3$, &c. that therefore

Its n^{th} Rank must be $=1+2+3+4+$ &c. $+:n-1:$

Also its $\overline{n-1}^{th}$ Rank $=1+2+3+4+$ &c. $+:n-2:$

Also its $\overline{n-2}^{th}$ Rank $=1+2+3+4+$ &c. $+:n-3:$

&c.

Therefore the n^{th} Rank of the lesser Series is (by substituting $:n-1:$ for n in the Canon inserted in *Theor.* II.) $= \frac{\overline{n-1}|^2+\overline{n-1}}{2}$: Also the $n-1^{th}$ Rank is $=\frac{\overline{n-2}|^2+\overline{n-2}}{2}$ Again the $n-2^{th}$ Rank is $=\frac{\overline{n-3}|^2+\overline{n-3}}{2}$: &c. Wherefore $\frac{\overline{n-1}|^2+\overline{n-1}}{2}+\frac{\overline{n-2}|^2+\overline{n-2}}{2}+\frac{\overline{n-3}|^2+\overline{n-3}}{2}$ + *&c.* is equal to the Sum of the lesser Series: But $\frac{1}{2}\times:$ $\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+$ *&c.* $+0:$ is $=\frac{z}{2}-\frac{n^2}{2}$; and $\frac{1}{2}\times:\overline{n-1}+\overline{n-2}+\overline{n-3}+$ &c. $+0:=\frac{nn-n}{4}:$ Wherefore the Sum of the lesser Series is $=\frac{z}{2}-\frac{nn}{2}+\frac{nn-n}{4}$: Consequently the Difference of the Sums of the greater and lesser Series is equal to $\frac{n^3+nn}{2}-\frac{z}{2}+\frac{nn}{2}-\frac{nn-n}{4}=z$ (by Supposition). Which Equation, being reduc'd, gives $z=\frac{n^3+1\frac{1}{2}nn+\frac{1}{2}n}{3}$. *Q. E. D.*

THEOREM IV.

In a Series of Cubes whose Roots are in an Arithmetical Progression increasing whose first Term = common Excess is 1, and Number of Terms $=n$ (as 1^3, 2^3, 3^3, 4^3, &c. n^3) I say that $\frac{n^4+2n^3+nn}{4}$ is equal to their Sum z, that is $= 1+8+27+64+$ &c. $+n^3$.

DEMONSTRATION.

The Sum z is manifestly equal to the Difference of the Sums of the greater and lesser following Series, each of whose Ranks is the Sum of the Squares of an ∸, *viz.*

$$=\left\{\begin{array}{l} \textit{Greater Series.} \\ 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \\ \overbrace{\qquad\qquad\qquad\qquad\qquad}^{} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\} - \left\{\begin{array}{l} \textit{Lesser Series.} \\ 0 \\ 1 \\ 1+4 \\ 1+4+9 \\ \overbrace{\qquad\qquad}^{} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\}$$

Difference of the two foregoing Series.

$$=\left\{\begin{array}{r} 1+4+9+16+\&c.+n^2 \\ 4+9+16+\&c.+n^2 \\ 9+16+\&c.+n^2 \\ 16+\&c.+n^2 \\ \overbrace{\qquad\qquad\qquad\qquad\qquad}^{} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\}$$

For $1=1^3$, $4+4=2^3$, $9+9+9=3^3$, &c. And $1+4+9+16+\&c.+n^2$ is (by *Theorem* III.) $=\frac{2n^3+3nn+n}{6}$; wherefore the Sum of the foregoing greater Series (being equal to n times: $1+4+9+16+\&c.+n^2$:) is $=\frac{2n^4+3n^3+nn}{6}$.

Again, in order to find the Sum of the lesser Series, consider, 0 being the first Rank, 1 the second Rank, $1+2^2=$ the third Rank, $1+2^2+3^2=$ the fourth Rank, &c. of the lesser Series; that therefore the

n^{th}

nth Rank thereof will be $=1+2^2+3^2+4^2+$ &c. $+\overline{n-1}|^2$
also the $\overline{n-1}^{\text{th}}$ Rank $= 1+2^2+3^2+4^2+$ &c. $+\overline{n-2}|^2$
again the $\overline{n-2}^{\text{th}}$ Rank $=1+2^2+3^2+4^2+$ &c. $+\overline{n-3}|^2$.
&c.

Therefore the n^{th} Rank of the lesser Series is (by *Theorem* III.) $=\frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{6}$: Also the $\overline{n-1}^{\text{th}}$ Rank is $=\frac{2\overline{n-2}|^3+3\overline{n-2}|^2+\overline{n-2}}{6}$: Again the $\overline{n-2}^{\text{th}}$ Rank is $=\frac{2\overline{n-3}|^3+3\overline{n-3}|^2+\overline{n-3}}{6}$: &c. Therefore $\frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{6}+\frac{2\overline{n-2}|^3+3\overline{n-2}|^2+\overline{n-2}}{6}+\frac{2\overline{n-3}|^3+3\overline{n-3}|^2+\overline{n-3}}{6}$ + &c. is equal to the Sum of the lesser Series: But $\frac{2}{6}\times : \overline{n-1}|^3+\overline{n-2}|^3+\overline{n-3}|^3+$ &c. $+0$: is $=\frac{z}{3}-\frac{n^3}{3}$; also $\frac{3}{6}\times : \overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+$ &c. $+0$: is (by *Theorem* III.) $=\frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{12}=\frac{2n^3-3nn+n}{12}$: And $\frac{1}{6}\times : \overline{n-1}+\overline{n-2}+\overline{n-3}+$ &c. $+0$: is (by *Theorem* II.) $=\frac{nn-n}{12}$. Wherefore the Sum of the lesses Series is $=\frac{z}{3}-\frac{n^3}{3}+\frac{2n^3-3nn+n+nn-n}{12}=\frac{z^3}{3}-\frac{n^3+nn}{6}$: Wherefore the Difference of the Sums of the greater and lesser Series is $=\frac{2n^4+3n^3+nn}{6}-\frac{z}{3}+\frac{n^3+nn}{6}=z$ (by Supposition): Which Equation, being reduc'd, gives $z=\frac{n^4+2n^3+nn}{4}$. *Q. E. D.*

THEOREM V.

In a Series of Biquadrats, whose Roots are in an Arithmetical Progression increasing, whose first Term is $= 1 =$ common Excess, and Number of Terms $= n$ (as 1^4, 2^4, 3^4, 4^4, &c. n^4); I say that $\frac{n^5 + 2\frac{1}{2}n^4 + 1\frac{2}{3}n^3 * - \frac{1}{6}n}{5}$ is $=$ to their Sum z.

Demonstration.

The Sum z is manifestly equal to the Difference of the Sums of the two next following Series, each of whose Ranks is the Sum of the Cubes of an $\div$, *viz.*

$$= \left\{ \begin{array}{l} \quad\quad \textit{Greater Series.} \\ 1+8+27+64+\&c.+n^3 \\ 1+8+27+64+\&c.+n^3 \\ 1+8+27+64+\&c.+n^3 \\ \underbrace{1+8+27+64+\&c.+n^3}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\} - \left\{ \begin{array}{l} \textit{Lesser Series.} \\ 0 \\ 1 \\ 1+8 \\ \underbrace{1+8+27}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\}$$

Difference of the two foregoing Series.

$$= \left\{ \begin{array}{r} 1+8+27+64+\&c.+n^3 \\ 8+27+64+\&c.+n^3 \\ 27+64+\&c.+n^3 \\ 64+\&c.+n^3 \\ \underbrace{\qquad\qquad\qquad\qquad\qquad}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\}$$

For

For $1=1^4$, $8+8=2^4$, $27+27+27=3^4$, &c. And $1+8+27+64+$ &c. $+n^3$ is (by *Theorem* IV.) $=\frac{n^4+2n^3+n^2}{4}$; therefore the Sum of the foregoing greater Series (being $=n$ times: $1+8+27+64+$ &c. $+n^3$:) is $=\frac{n^5+2n^4+n^3}{4}$.

Next, to find the Sum of the lesser Series, consider, since 0 is the first Rank thereof, 1 the second Rank, $1^3+2^3=$ the third Rank, $1^3+2^3+3^3=$ the fourth Rank, *&c.* that

$1^3+2^3+3^3+4^3+$ &c. $+\overline{n-1}|^3$ will be $=n^{th}$ Rank; also
$1^3+2^3+3^3+4^3+$ &c. $+\overline{n-2}|^3=$ the $\overline{n-1}^{th}$ Rank; also
$1^3+2^3+3^3+4^3+$ &c. $+\overline{n-3}|^3=$ the $\overline{n-2}^{th}$ Rank
&c. of the lesser Series:

Wherefore the *n*th Rank thereof is (by *Theorem* IV.) $=\frac{\overline{n-1}|^4+2\overline{n-1}|^3+\overline{n-1}|^2}{4}$: Also the $\overline{n-1}^{th}$ Rank $=\frac{\overline{n-2}|^4+2\overline{n-2}|^3+\overline{n-2}|^2}{4}$: Again the $\overline{n-2}^{th}$ Rank $=\frac{\overline{n-3}|^4+2\overline{n-3}|^3+\overline{n-3}|^2}{4}$: &c. Therefore $\frac{1}{4}\times:\overline{n-1}|^4+\overline{n-2}|^4+\overline{n-3}|^4+$ &c. $+0:+\frac{2}{4}\times:\overline{n-1}|^3+\overline{n-2}|^3+\overline{n-3}|^3+$ &c. $+0:+\frac{1}{4}\times:\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+$ &c. $+0$: is $=$ the Sum of the lesser Series: But $\frac{1}{4}\times:\overline{n-1}|^4+\overline{n-2}|^4+\overline{n-3}|^4+$ &c. $+0$: is $=\frac{z}{4}-\frac{n^4}{4}$:

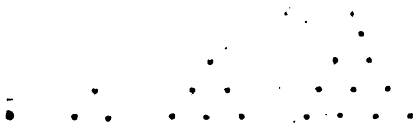
Also $\frac{2}{4}\times:\overline{n-1}|^3+\overline{n-2}|^3+\overline{n-3}|^3+$ &c. $+0$: is (by *Theorem* IV.) $=\frac{2}{16}\times:\overline{n-1}|^4+2\overline{n-1}|^3+\overline{n-1}|^2:=\frac{n^4-2n^3+n^2}{8}$: Also $\frac{1}{4}\times:\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+$ &c. $+0$: is (by *Theorem* III.) $=\frac{1}{24}\times:2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}:=\frac{2n^3-3nn+n}{24}$: Wherefore the Sum of the lesser Series is $=\frac{z}{4}-\frac{n^4}{4}+\frac{n^4-2n^3+nn}{8}+$

$\frac{2n^3 - 3nn + nn}{24}$. Consequently the Difference of the Sums of the greater and lesser Series is $= \frac{n^5 + 2n^4 + nnn}{4} - \frac{z}{4} + \frac{n^4}{4} - \frac{n^4 - 2n^3 + nn}{8} - \frac{2n^3 - 3nn + n}{24} = \frac{6n^5 + 15n^4 + 10n^3 * - n}{24} - \frac{z}{4} = z$ (by Supposition) : And by transposing $\frac{z}{4}$, and then multiplying each Part by $\frac{4}{5}$, we shall have $z = \frac{n^5 + 2\frac{1}{2}n^4 + 1\frac{2}{3}n^3 - \frac{1}{6}n}{5}$. *Q. E. D.*

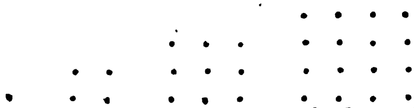
PART XIV.

Of Polygonal Numbers.

POlygonal, or Multangular Numbers are the Sums or Aggregates of a Rank of Numbers in Arithmetical Progression continued from Unity, and are so called, because they represent the Number of Points that are required to fill such regular Polygons at equal Distances or Lines drawn parallel to the Sides of the Figure; as



&c. are Triangulars.



&c. are Quadrangulars.

&c. And are thus form'd:

Num. in Arit. Progression.	*Whose com. Differ. is*	*Sums added from* 1.	*Polygon.*
1, 2, 3, 4, &c.	1	1, 3, 6, 10, 15, 21, &c.	Triang.
1, 3, 5, 7, &c.	2	1, 4, 9, 16, 25, 36, &c.	Quadr.
1, 4, 7, 10, &c.	3	1, 5, 12, 22, 35, 51, &c.	Pentang.
1, 5, 9, 13, &c.	4	1, 6, 15, 28, 45, 66, &c.	Hexang.
&c.	&c.	&c.	&c.

Hence, by Inspection, these two Observations are evident;

1. That the common Difference will be always less by 2, than the Number of Angles.
2. That the Side of the Polygon is equal to the Number of Terms which compose it.

There-

Therefore, putting $d =$ Common Difference.

$n =$ Number of Terms.

$p =$ Polygon.

We have $p = \frac{nn - n}{2} \times d + n$.

Demonstration.

Since every Polygon is the Aggregate of Numbers in $\div$, whose first Term is 1; therefore (by *Part VIII. Chap.* 1. *Step* 17.) p is $= \frac{nn - n}{2} d + n$. *Q. E. D.*

Corollary.

And because d is given (by *Observation* 1.) if its Value be substituted in this general Theorem, we may deduce particular ones for each Polygon: As in

$$\left.\begin{matrix}\text{Triang.}\\ \text{Quadrang.}\\ \text{Pentang.}\\ \text{Hexang.}\end{matrix}\right\} d \text{ is} = \left\{\begin{matrix}1\\ 2\\ 3\\ 4\end{matrix}\right\} \text{and } p = \frac{nn - n}{2} d + n \because p = \left\{\begin{matrix}\frac{nn + n}{2}\\ nn\\ \frac{3nn - n}{2}\\ 2nn - n\end{matrix}\right.$$

But, because n may not be always given, since we find $p = \frac{nn - n}{2} d + n$; n therefore will be found (by *Part* X.) $= \frac{d - 2 + \sqrt{: dd - 4d + 4 + 8dp :}}{2d}$: And because d is given (by *Obser.* 1.) if we substitute its Value, we shall have particular Theorems in this Case likewise; *viz.* in a

Triang.

$$\left.\begin{array}{l}\text{Triang.}\\[2ex]\text{Quadrang.}\\[2ex]\text{Pentang.}\\[2ex]\text{Hexang.}\end{array}\right\} d=\left\{\begin{array}{c}1\\[2ex]2\\[2ex]3\\[2ex]4\end{array}\right\} \because n=\left\{\begin{array}{l}\dfrac{-1+\sqrt{:1+8p:}}{2}\\[2ex]\dfrac{0+\sqrt{:0+16p:}}{4}\\[2ex]\dfrac{1+\sqrt{:1+24p:}}{6}\\[2ex]\dfrac{2+\sqrt{:4+32p:}}{8}\\ \text{\&c.}\end{array}\right.$$

Thus having shew'd how from the Side given to find the Polygon; or from the Polygon to find the Side; I shall now gsve an universal Theorem for finding any Trigonal, Pyramidal, or other Number undermentioned: As

								Or the Figurate Number of the	
Units.	1,	1,	1,	1,	1,	1		1st	Order.
Laterals.	1,	2,	3,	4,	5,	6		2d	Order.
Trigonals.	1,	3,	6,	10,	15,	21		3d	Order.
Pyramid.	1,	4,	10,	20,	35,	56	&c.	4th	Order.
2d Pyramid.	1,	5,	15,	35,	70,	126		5th	Order.
3d Pyramid.	1,	6,	21,	56,	126,	252		6th	Order.
			&c.					&c.	

Observation I.

Here it is evident, that each figurate Number is the Aggreate of the preceding Series so far, or of the preceding figurate Number and that above itself.

Observation II.

It is also evident, if b and c be suppos'd equal to any two whole Numbers, that b figurate Number of c Order is equal to c figurate Number of b Order.

LEMMA.

If n be $=$ the Side of a figurate Number of such an Order, that 1, $1\times n$, $1\times\frac{n}{1}\times\frac{n+1}{2}$, $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$,

1 ×

$1 \times \frac{n+0}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, &c. *i. e.* (since $\frac{a}{e} \times \frac{z}{y}$ is $= \frac{z}{e} \times \frac{a}{y}$) 1, $1 \times n$, $1 \times \frac{n+1}{1} \times \frac{n}{2}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, &c. be equal to the respective figurate Numbers of that Order; And, if m be $= n+1 =$ the Side of the figurate Number of the next greater Order; I say that 1, $1 \times m$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2}$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{m+3}{4}$, &c. shall be equal to the respective figurate Numbers of that Order whose Side is m.

Demonstration.

It is evident (by the Nature of Multiplication) that $1 = 1$, $1 \times n+1 = 1 \times n+1$, $1 \times \frac{n+1}{1} \times \frac{n}{2} + 1 \times n+1 = 1 \times \frac{n+1}{1} \times \overline{\frac{n}{2}+1} \left(\frac{n+2}{2}\right)$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3} + 1 \times \frac{n+1}{1} \times \frac{n+2}{2} = 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \overline{\frac{n}{3}+1} \left(\frac{n+3}{3}\right)$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4} + 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} = 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \overline{\frac{n}{4}+1} \left(\frac{n+4}{4}\right)$, &c: Therefore, if 1, n, $1 \times \frac{n+1}{1} \times \frac{n}{2}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, &c.

be

be equal to the respective figurate Numbers of the Order whose Side is $= n$; then $1, 1 \times n+1, 1 \times \frac{n+1}{1} \times \frac{n+2}{2}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n+4}{4}$, &c. will be equal to the respective figurate Numbers of the Order whose Side is $= m$ (by the Nature of figurate Numbers), that is equal to

$1, 1 \times m, 1 \times \frac{m}{1} \times \frac{m+1}{2}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{m+3}{4}$, &c. respectively (since m is $= n+1$). *Q. E. D.*

Scholium I.

It is evident that, if n be $=$ the Side of a figurate Number of the first Order, or that of Units; then n will be $= 1$, and $1, 1 \times n, 1 \times \frac{n}{1} \times \frac{n+1}{2}, 1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. will be equal to $1, 1 \times 1, 1 \times 1 \times 1, 1 \times 1 \times 1 \times 1$, &c. respectively; that is, each figurate Number of that Order will be 1, which are manifestly the respective figurate Numbers of that Order: Therefore (by the precedent *Lemma*) $1, 1 \times m, 1 \times \frac{m}{1} \times \frac{m+1}{2}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, &c. ($m$ being always $= n+1$) are equal to the respective figurate Numbers of the 2^d Order, or that of Laterals: And, if n be $=$ the Side of a figurate Number of the 2^d Order; then, since $1, 1 \times n, 1 \times n \times \frac{n+1}{2}, 1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are (by this *Scholium*) equal to the respective figurate Numbers of the 2^d Order, $1, 1 \times m, 1 \times m \times \frac{m+1}{2}, 1 \times m \times \frac{m+1}{2} \times$

$\frac{m+2}{3}$, &c. will be (by the preceding *Lemma*) equal to the respective figurate Numbers of the 3d Order: Again, if n be $=3$ the Side of a figurate Number of the 3d Order, then (since $1, 1 \times n, 1 \times \frac{n}{1} \times \frac{n+1}{2}, 1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are equal to the respective figurate Numbers of the 3d Order), $1, 1 \times m, 1 \times \frac{m}{1} \times \frac{m+1}{2}, 1 \times \frac{m+0}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, &c. will be (by our *Lemma*) equal to the respective figurate Numbers of the 4th Order: And so on. Whence may be deduc'd the following

Corollary I.

If n be $=$ the Side of a figurate Number of any Order; then $1, 1 \times n, 1 \times n \times \frac{n+1}{2}, 1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}, 1 \times \frac{n+0}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, &c. will be $=$ to the 1st, 2d, 3d, 4th, 5th, &c. figurate Numbers of that Order respectively; which is the universal Theorem I promis'd to give.

Corollary II.

Since n being suppos'd $=$ the Side of a figurate Number of any Order, $1, 1 \times n, 1 \times \frac{n}{1} \times \frac{n+1}{2}, 1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are equal to the 1st, 2d, 3d, 4th, &c. figurate Numbers of that Order respectively; therefore, the following n, being suppos'd $=$ the Number of Terms, $1, 1 \times n, 1 \times n \times \frac{n+1}{2}, 1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. shall be (by

Obser-

Observation 2.) equal to the n^{th} figurate Number of the 1st, 2d, 3d, 4th, &c. Order respectively.

Scholium II.

Hence also may be had the Sums of the Powers of an ∺, whose first Term and common Excess are each $=1$; thus, supposing

§. I. $n=$ the Number of Terms, $1\times n\times\frac{n+1}{2}=\frac{nn+n}{2}$ will be (by *Obs.* 1. and 2d *Coroll.*) $=$ Sum of the 2d Series, or of the Series of Laterals to the nth Term inclusive, or $=$ the nth figurate Number of the 3d Order;

Therefore (by *Observ.* 1.) $\frac{1^2+1}{2}+\frac{2^2+2}{2}+\frac{3^2+3}{2}+\frac{4^2+4}{2}+$ &c. to n Places continued is $=$ the Sum of the figurate Numbers of the 3d Order to the nth Term inclusive, or $=$ the nth figurate Number of the 4th Order $=1\times n\times\frac{n+1}{2}\times\frac{n+2}{3}$ (by *Coroll.* 2.) $=\frac{n^3+3nn+2n}{6}$: But $\frac{1}{2}\times:1+2+3+4+$ &c. to n Places, or Terms continued: is (by §. 1.) $=\frac{nn+n}{2\times 2}$: Wherefore $\frac{1}{2}\times:1^2+2^2+3^2+4^2+$ &c. to n Terms continued: is $=\frac{n^3+3nn+2n}{6}-\frac{nn+n}{4}=\frac{n^3+\frac{3}{2}n^2+\frac{1}{2}n}{6}$; consequently

II. $1^2+2^2+3^2+4^2+5^2+$ &c. to n Terms continued, or the Sum of the Squares of an ∺ whose first Term and common Excess are each equal to 1, and Number of Terms equal to n is $=\frac{n^3+1\frac{1}{2}nn+\frac{1}{2}n}{3}$.

Again, since $\frac{n^3+3n^2+2n}{6}$ is $=$ the nth figurate Number of the 4th Order; therefore $\frac{1^3+3\times 1^2+2\times 1}{6}+$

 2^3

$\frac{2^3+3\times 2^2+2\times 2}{6}+\frac{3^3+3\times 3^2+2\times 3}{6}+\frac{4^3+3\times 4^2+2\times 4}{6}+$ &c. to n Ploces continued is (by *Obs.* 1.) $=$ to the Sum of the figurate Numbers of the 4th Order to n Places continued; or $=$ nth figurate Number of the 5th Order, which is (by *Coroll.* 2.) $= 1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}$. But $\frac{1}{6}\times : 1^2+2^2+3^2+4^2+$ &c. to n Terms continued: is (by §. II.) $=\frac{1}{6}\times\frac{n^3+1\frac{1}{2}nn+\frac{1}{2}n}{3}$: And $\frac{1}{6}\times : 1+2+3+4+$ &c. to n Terms continued: $=\frac{1}{6}\times\frac{nn+n}{2}$ (*per* §. 1.); wherefore $\frac{1}{6}\times : 1^3+2^3+3^3+4^3+$ &c. to n Terms continued: is $=\frac{n^4+6n^3+11n^2+6n}{24}-\frac{n^3+1\frac{1}{2}nn+\frac{1}{2}n}{6}-\frac{nn+n}{6}=\frac{n^4+2n^3+n^2}{24}$; consequently

III. $1^3+2^3+3^3+4^3+5^3+$ &c. to n Terms continued, or the Sum of the Cubes of an ∸ whose first Term and common Excess are each equal to 1, and Number of Terms equal to n, is $=\frac{n^4+2n^3+nn}{4}$.

Again, since $\frac{n^4+6n^3+11n^2+6n}{24}$ is $=$ the n^{th} figurate Number of the 5th Order, it is evident (by *Obs.* 1.) that $\frac{1^4+6\times 1^3+11\times 1^2+6\times 1}{24}+\frac{2^4+6\times 2^3+11\times 2^2+6\times 2}{24}+\frac{3^4+6\times 3^3+11\times 3^2+6\times 3}{24}+\frac{4^4+6\times 4^3+11\times 4^2+6\times 4}{24}+$ &c. continued to n

Places

Places is = the Sum of the figurate Numbers of the 5th Order to n Places continued = to the nth figurate Number of the 6th Order, which is (by *Coroll.* 2.) $= 1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} = \frac{n^5 + 10n^4 + 35n^3 + 50n^2 + 24n}{120}$. But $\frac{6}{24} \times : 1^3 + 2^3 + 3^3 + 4^3 +$ &c. to n Terms continued : is (by §. III.) $= \frac{6}{24} \times \frac{n^4 + 2n^3 + nn}{4} = \frac{7\frac{1}{2}n^4 + 15n^3 + 7\frac{1}{2}nn}{120}$; also $\frac{11}{24} \times : 1^2 + 2^2 + 3^2 + 4^2 +$ &c. to n Terms continued : $= \frac{11}{24} \times \frac{n^3 + \frac{3}{2}nn + \frac{1}{2}n}{3}$ (by § II.) $= \frac{18\frac{1}{3}n^3 + 27\frac{1}{2}nn + 9\frac{1}{6}n}{120}$; and $\frac{6}{24} \times : 1 + 2 + 3 + 4 +$ &c. to n Terms continued : is (by §. I.) $= \frac{6}{24} \times \frac{nn+n}{2} = \frac{15nn + 15n}{120}$: Wherefore $\frac{1}{24} \times : 1^4 + 2^4 + 3^4 + 4^4 +$ &c. to n Terms continued : is $= \frac{n^5 + 10n^4 + 35n^3 + 50nn + 24n}{120} - \frac{7\frac{1}{2}n^4 + 15n^3 + 7\frac{1}{2}nn}{120} - \frac{18\frac{1}{3}n^3 + 27\frac{1}{2}nn + 9\frac{1}{6}n}{120} - \frac{15nn + 15n}{120} = \frac{n^5 + 2\frac{1}{2}n^4 + 1\frac{2}{3}n^3 * - \frac{1}{6}n}{120}$; consequently

IV. $1^4 + 2^4 + 3^4 + 4^4 +$ &c. to n Terms continued, or the Sum of the Biquadrates of an ÷ whose first Term and common Excess are each equal to 1, and Number of Terms equal to n, is $= \frac{n^5 + 2\frac{1}{2}n^4 + 1\frac{2}{3}n^3 * - \frac{1}{6}n}{5}$.

Corollary III.

If p be $=$ any Affirmative whole Number greater than 1, and $n =$ to an * indefinite Number; then $1^p + 2^p + 3^p + 4^p + 5^p +$ &c. to n Terms continued is $= \frac{n^{p+1}}{p+1} + \frac{1}{2} \times n^p \multimap$. Canon.

* *Indefinitely great.*

NB. Tho' the 2d Term of this Canon namely $\frac{1}{2} \times n^p$ be indefinitely leſs than the firſt, *viz.* $\frac{n^{p+1}}{p+1}$, yet it will be convenient to have it inſerted for the great Uſe we have of that Canon (which will appear in *Book* II.): And ſince the third Term is indefinitely leſs than the ſecond, and the 4th Term indefinitely leſs than the 3d, *&c.* it will be needleſs to inſert the 3d, 4th, *&c.* Terms in it only by $\multimap$; and therefore the ſaid Canon will ſtand after the moſt convenient Manner for our Uſe, as we have above deſign'd it.

PART

PART XV.

Of the Nature of Series, or of Approximations.

CHAP. I.

The Demonstration of Sir *If. Newton*'s Theorem.

LEMMA.

IF n be $=$ fuch a Number, that $1, n, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. be equal to the Unciæ of the nth Power of a Binomial or Refidual; and if m be $= n+1$; I fay that $1, m, \frac{m}{1} \times \frac{m-1}{2}, \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}, \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c. fhall be equal to the Unciæ of the mth Power of the Binomial or Refidual.

Demonftration.

It is evident by the Genefis of Powers [*See Pages* 35, 36 and 37] If $1, n, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. be equal to the Unciæ of the nth Power of a Binomial or Refidual, that the Unciæ of the $n+1^{th}$ Power of a Binomial or Refidual will be equal to $1, n+1, \frac{n}{1} \times \frac{n-1}{2} + n, n \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times$

$n-1$

$\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c.

equal to 1, $n+1$, $\frac{n}{1}\times\overline{\frac{n-1}{2}+1}\left(\frac{n+1}{2}\right)$, $\frac{n}{1}\times\frac{n-1}{2}\times\overline{\frac{n-2}{3}+1}\left(\frac{n+1}{3}\right)$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\overline{\frac{n-3}{4}+1}\left(\frac{n+1}{4}\right)$, &c. respectively (by the Nature of Multiplication)

equal to 1, $n+1$, $\frac{n+1}{1}\times\frac{n}{2}$, $\frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}$, $\frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}$, &c. respectively (since a, e, z and y, being equal to any four Quantities, $\frac{a}{e}\times\frac{z}{y}$ is $=\frac{z}{e}\times\frac{a}{y}$)

equal to 1, m, $\frac{m}{1}\times\frac{m-1}{2}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, *&c.* respectively (since m is $=n+1$);

And that they will be so *in infinitum*, sufficiently appears from the Nature of the Operation. *Q. E. D.*

Scholia.

If n be $=1$ the Index of the Root, and $m=n+1=2$ the Index of the Square; then, since 1, $n\ (=1)$, $\frac{n}{1}\times\frac{n-1}{2}\ (=0)$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\ (=0)$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\ (=0)$, &c. are $=$ to the Unciæ of the Root of a Binomial or Residual, 1, m, $\frac{m}{1}\times\frac{m-1}{2}$, $\left(\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\right.$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c.) will be (by our *Lemma*) equal to the Unciæ of the Square of a Binomial or Residual.

Again,

Again, if n be $= 2$ the Index of the Square, and $m = n + 1 = 3$ the Index of the Cube; then, since 1, $n\,(=2)$, $\frac{n}{1} \times \frac{n-1}{2}\,(=1)$, $(\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}\,(=0)$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}\,(=0)$, &c.) are equal to the Unciæ of the Square of a Binomial, or Residual, 1, m, $\frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $(\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c.) will be (by our *Lemma*) equal to the Unciæ of the Cube of a Binomial or Residual.

Again, supposing $n = 3$ the Index of the Cube; then, since (by what has been already said) 1, n, $\frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $(\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.) are equal to the Unciæ of the Cube of a Binomial or Residual, 1, m, $\frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, (&c.) will be (by our *Lemma*) equal to the Unciæ of the 4th Power of a Binomial or Residual. *&c.* Hence

I. If n be $=$ any affirmative whole Number, the nth Power of any Binomial or Residual $a \pm x$ is $= a^n + n a^{n-1} \times \pm x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} \times + x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} \times \pm x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} \times + x^4$, &c.

Again, if n and m be equal to any affirmative whole Numbers, then $\overline{a \pm x}|^n$ is (by what has been before said)

 $= a$

$= a^n \pm n a^{n-1} x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} x^2$, &c; and

$\overline{a \pm x}|^m = a^m \pm m a^{m-1} x + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} x^2$, &c:

And $\overline{a \pm x}|^n \times \overline{a \pm x}|^m = \overline{a \pm x}|^{n+m} = a^{n+m} \pm \overline{n+m}$
$a^{n+m-1} x + \overline{n+m} \times \frac{n+m-1}{2} a^{n+m-2} x^2$, &c.

Whence: $a^m \pm m a^{m-1} x + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} x^2$, &c;

Multipl. by: $a^n \pm n a^{n-1} x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} x^2$, &c:

Produces $a^{m+n} \pm \overline{m+n}\, a^{m+n-1} x + \frac{m+n}{1} \times \frac{m+n-1}{2}$
$a^{m+n-2} x^2$, &c.

The Truth of this will appear from Algebraic Multiplication as far as you are pleased to continue the Operation; and that it will be so *in infinitum* is manifest from what has been already said.

☞ Now, since this Product is such as I have express'd, it must be so, altho' n and m were equal to any Numbers whatsoever; for this Multiplication does not distinguish what Numbers they are equal to; but, on the contrary, being Symbols, are to be consider'd therein only as such, that is, as universal. Hence

II. $\overline{a \pm x}|^{\frac{1}{n}}$ is $= a^{\frac{1}{n}} \pm \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} a^{\frac{1}{n}-2}$
$x^2 \pm \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} a^{\frac{1}{n}-3} x^3$, &c. *in infinitum*; for this multiplied by itself is $(= \overline{a \pm x}|^{\frac{2}{n}}) = a^{\frac{2}{n}} \pm \frac{2}{n}$
$a^{\frac{2}{n}-1} x + \frac{\frac{2}{n}}{1} \times \frac{\frac{2}{n}-1}{2} a^{\frac{2}{n}-2} x^2$, &c. (by what is above said); and this multiplied by $\overline{a \pm x}|^{\frac{1}{n}}$ is $(= \overline{a \pm x}|^{\frac{3}{n}}) =$
$a^{\frac{3}{n}} \pm \frac{3}{n} a^{\frac{3}{n}-1} x + \frac{\frac{3}{n}}{1} \times \frac{\frac{3}{n}-1}{2} a^{\frac{3}{n}-2} x^2$, &c. and there-

fore

fore $\overline{a^{\frac{1}{n}} \pm \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} a^{\frac{1}{n}-2} x^2, \&c.}\Big|^{n}$ is $=$

$a^{\frac{n}{n}} \pm \frac{n}{n} a^{\frac{n}{n}-1} x + \frac{\frac{n}{n}}{1} \times \frac{\frac{n}{n}-1}{2} a^{\frac{n}{n}-2} x^2$, &c. that is $= a \pm x$.

Hence it is manifest that m and n being equal to affirmative Numbers,

III. $\overline{a \pm x}\big|^{\frac{m}{n}}$ is $= a^{\frac{m}{n}} \pm \frac{m}{n} a^{\frac{m}{n}-1} x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} a^{\frac{m}{n}-2} x^2$, &c; wherefore

IV. $\overline{a \pm x}\big|^{-\frac{m}{n}}$ is $= a^{-\frac{m}{n}} \pm \frac{-m}{n} a^{-\frac{m}{n}-1} x + \frac{-\frac{m}{n}}{1} \times \frac{-\frac{m}{n}-1}{2} a^{-\frac{m}{n}-2} x^2$, &c; for this multiplied by $\overline{a \pm x}\big|^{\frac{m}{n}}$, when evolved as above, is (by what was before said) $= a^{\frac{m}{n}-\frac{m}{n}} \pm \frac{m}{n} - \frac{m}{n} \times a^{\frac{m}{n}-\frac{m}{n}-1} x$, &c. $=$

$a^{\frac{m}{n}-\frac{m}{n}} \pm 0 = 1$, that is $= \overline{a \pm x}\big|^{\frac{m}{n}-\frac{m}{n}}$.

Corollary.

If any Binomial or Residual $a \pm x$ be rais'd to any Power whose Index is n (n representing any Number whatsoever) the nth Power of that Binomial or Residual will be $a^n +$

$n a^{n-1} \times \pm x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} \times + x^2 + \frac{n}{1} \times \frac{n-1}{2}$

$\times \frac{n-2}{3} a^{n-3} \times \pm x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$

$a^{n-4} x^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5} \times$

$\pm x^5$, &c.

Or, putting $q = \frac{x}{a}$, then $a \pm x = a \pm a \times \frac{x}{a} = a \pm aq \; (= a \times \overline{1 \pm q})$; and $\overline{a \pm aq}|^n$ is $= a^n \pm n a^n q + \frac{n}{1} \times \frac{n-1}{2} a^n q^2 \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^n q^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^n q^4$, &c.

Or, putting A = 1st Term, B = 2d, C = 3d, D = 4th, &c; then it will be $\overline{a \pm aq}|^n = a^n + n A \times \pm q + \frac{n-1}{2} B \times \pm q + \frac{n-2}{3} C \times \pm q + \frac{n-3}{4} D \times \pm q$, &c.

If it were required to raise a Trinomial, Quadrinomial, &c. or an Infinitomial to any given Power, it may be done by the foregoing *Corollary*: As for instance, suppose the Infinitomial $az + bz^2 + cz^3 + dz^4 +$ &c. were to be raised to the Power whose Index is n; then

$\overline{az + bz^2 + cz^3 + dz^4 + \&c.}|^n$ is $= a^n z^n + n \overline{az}|^{n-1} \times : bz^2 + cz^3 + dz^4 + \&c : + \frac{n}{1} \times \frac{n-1}{2} \overline{az}|^{n-2} \times \overline{bz^2 + cz^3 + dz^4 + \&c.}|^2 +$ &c.

CHAP. II.

Of the Converging Series.

MR. *Raphson*'s Method in raising particular Theorems for extracting the Roots of all Equations whatsoever is as follows; *viz.* He supposes the Root in any Equation, which he calls a to be = a Binomial of g a known, and x an unknown Quantity, and then raises the Binomial $g + x$ to all the Powers of a in the propos'd Equation, and with *and* by these Values of a and its Powers he reduces the propos'd Equation to another exclusive of a. Then he rejects, or leaves

☞ *See Pages* 65, 66, *and* 67.

leaves out all the Members of the new Equation wherein there is any Power of x, exceeding the Root; and with the other Members he makes a ſuppos'd Equation, and thereby finds the ſuppos'd Value of x, or the Theorem required.

Now if this Method be applied to an univerſal Equation, an univerſal Theorem will thereby be raiſed which will include all his firſt Set of Theorems, and indeed all others of that Kind for extracting the Roots of all Equations whatſoever: Let us therefore ſuppoſe the propos'd Equation to be repreſented by the univerſal one

$$a^n \pm p\,a^{n-1} \pm q\,a^{n-2} \pm r\,a^{n-3} \pm \&c. = \pm N.$$

In which N is = the given abſolute Number.
n = the Exponent of the higheſt Power of a = the Root ſought. } *NB.* n muſt be not $\sqsupset$ 2.
1, p, q, r, &c. = to the reſpective Coefficients of the Powers of a in the propos'd Equation.

Let us alſo ſuppoſe $g+x=a$, which g is = the known Part of the Root ſought a, and ought to be taken as near the ſought Root as may be, whether it be greater or leſs than the ſaid Root, and x is = the unknown Part of the Root ſought, whoſe Value may be negative or affirmative, according as that of g is taken greater or leſs than the Truth:

Then the foregoing Univerſal Equation will become

$$\overline{g+x}|^{n} \pm p\times\overline{g+x}|^{n-1} \pm q\times\overline{g+x}|^{n-2} \pm r\times\overline{g+x}|^{n-3}$$
$$\pm \&c. = \pm N.$$

that is, by Sir *Iſ. Newton's* Theorem,

$$\overline{g+x}|^{n} = g^{n} + ng^{n-1}\,x + n\times\frac{n-1}{2}\,g^{n-2}\,x^2 + \&c.$$

$$\pm p\times\overline{g+x}|^{n-1} = \pm p\times: g^{n-1} + \frac{n-1}{1}\,g^{n-2}\,x + \frac{n-1}{1}\times\frac{n-2}{2}\,g^{n-3}\,x^2 + \&c:$$

$\pm q\times$

$$\left.\begin{array}{l}\pm q\times\overline{g+x}|^{n-2}=\pm q\times: g^{n-2}+\frac{n-2}{1}\\ \quad g^{n-3}x+\frac{n-2}{1}\times\frac{n-3}{2}g^{n-4}x^2+\&c:\\ \pm r\times\overline{g+x}|^{n-3}=\pm r\times: g^{n-3}+\frac{n-3}{1}\\ \quad g^{n-4}x+\frac{n-3}{1}\times\frac{n-4}{2}g^{n-5}x^2+\&c:\\ \pm\&c.=\pm\&c.\end{array}\right\}=\pm N.$$

Now, by rejecting all the Members in the last Equation wherein any Power of x, exceeding the Root, is contain'd, we have

$$\left.\begin{array}{ll} g^n & +\; ng^{n-1}x\\ \pm pg^{n-1} & \pm\overline{pn-p}g^{n-2}x\\ \pm qg^{n-2} & \pm\overline{qn-2q}g^{n-3}x\\ \pm rg^{n-3} & \pm\overline{rn-3r}g^{n-4}x\\ \pm\&c. & \end{array}\right\}=\pm N.$$

And by Transposition

$$ng^{n-1}x\pm\overline{pn-p}g^{n-2}x\pm\overline{qn-2q}g^{n-3}x\pm\overline{rn-3r}g^{n-4}x\pm\&c.=\pm N-g^n\mp pg^{n-1}\mp qg^{n-2}\mp rg^{n-3}\mp\&c.$$

And by Division we have

$$x=\frac{\pm N-g^n\mp pg^{n-1}\mp qg^{n-2}\mp rg^{n-3}\mp\&c.}{ng^{n-1}\pm\overline{pn-p}g^{n-2}\pm\overline{qn-2q}g^{n-3}\pm\overline{rn-3r}g^{n-4}\pm\&c.}$$

Which Theorem exhibits all possible particular ones for extracting Roots, according to the first sort of Mr. *Raphson*'s, agreeing exactly with them, as will be found on Trial; always

always remembring that the Signs in the Dividend muſt be contrary to thoſe in the Equation, and in the Diviſor the ſame reſpectively. And likewiſe it will be proper to take notice, that if any Term be wanting in the Equation, the ſame muſt be omitted in the Theorem. Thus,

If $a^n = N$, then $\left\{\begin{array}{l} p=0 \\ q=0 \\ r=0 \\ \text{\&c.} \end{array}\right\}$, and $x = \frac{N-g^n}{ng^{n-1}}$, which is an univerſal Canon or Theorem for extracting the Roots of all pure Powers whatſoever.

Now the Method of finding the affirmative Roots of any Equation whatſoever by the above univerſal Theorem is this;

First, Deduce from the univerſal Theorem the particular one for extracting the Root (or Roots) of the propos'd Equation; then ſuppoſe $g =$ ſome Number as near to the required Value of a as you can gueſs, and, by the particular Theorem, find the ſuppos'd Value of x, which add to the ſaid ſuppos'd Value of g, and call that Sum g, that is your 2[d] g; with which 2[d] g (by the ſaid particular Theorem) find another x, and call the Sum of the laſt found x and of g the 2d your 3d g; and ſo proceed 'till ſome g be found as near a, the true Root required, as ſhall be judged ſufficient.

That this Method of proceeding will give the true Value of a proxime in the foregoing univerſal ſimple Equation I demonſtrate, thus

The ſuppos'd Value of g muſt be either equal to the affirmative Value of a (for it is manifeſt it hath but one Value that is affirmative, which is what we ſeek), or leſs than it, or greater.

Caſe 1. If it be equal to it (that is, if $g=a$), then the Theorem we are about demonſtrating, to wit $x = \frac{N-g^n}{ng^{n-1}}$ will become $x = \frac{N-a^n}{na^{n-1}}$; that is to ſay $x = 0$, as it muſt, in that Caſe, evidently be.

Caſe

Case 2. If the Value of g be taken less than that of a; then, I say the Value of the 2^d g will be greater than that of a; For $2^d\, g = g + \frac{N - g^n}{n g^{n-1}}$; and $a = g + \frac{N - g^n - : n}{n g^{n-1}}$ $\times \frac{n-1}{2} g^{n-2} x^2$, &c:

But, since the Value of g (which is suppos'd to be affirmative) is less than that of a, the true Value of x will be affirmative; consequently (n being suppos'd not $\sqsupset 2$)

$$\frac{N - g^n}{n g^{n-1}} \sqsubset \frac{N - g^n - : n \times \frac{n-1}{2} g^{n-2} x^2, \&c:}{n g^{n-1}}$$

Therefore $g + \frac{N - g^n}{n g^{n-1}}$ $(= 2^d g) \sqsubset$

$$g + \frac{N - g^n - : n \times \frac{n-1}{2} g^{n-2} x^2, \&c:}{n g^{n-1}} (= a)$$

Case 3. If the Value of any g be greater than that of a; I say the Value of the next following g will still be $\sqsubset a$, but $\sqsupset$ the foregoing g; for

$$\text{Following } g = g + \frac{N - g^n}{n g^{n-1}}$$

$$\text{And } a = g + \frac{N - g^n - n \times \frac{n-1}{2} g^{n-2} x^2 - \frac{n}{1} \times \frac{n-1}{2}}{n g^{n-1}}$$

$\times \frac{n-2}{3} g^{n-3} x^3$, &c.

But, since g is $\sqsubset a$, the true Value of x is Negative; let us therefore design it here by a negative Symbol, *viz* $-z$; then it is evident that

$g-z$

$g-z$ is $\sqsubset$ $g-z-\frac{n-1}{2}g^{-1}z^2-$ * &c. *sine fine*

that is $g-z$ is $\sqsubset$ $\overline{g^n-ng^{n-1}z}|^{\frac{1}{n}}$; wherefore $\overline{g-z}|^n$ $=g^n-ng^{n-1}z+n\times\frac{n-1}{2}g^{n-2}z^2-n\times\frac{n-1}{2}\times\frac{n-2}{3}g^{n-3}z^3$, &c. is $\sqsubset$ $g^n-ng^{n-1}z$: Whence $\frac{n}{1}\times\frac{n-1}{2}g^{n-2}z^2-\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}g^{n-3}z^3$, &c. is $\sqsubset$ o.

And by substituting $+x$ for $-z$ we have $\frac{n}{1}\times\frac{n-1}{2}g^{n-2}x^2+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}g^{n-3}x^3$, &c. $\sqsubset$ o;

Consequently $g+\frac{N-g^n}{ng^{n-1}}$ $\left(=\text{following } g\right)$ is $\sqsubset$ $g+\frac{N-g^n-\frac{n}{1}\times\frac{n-1}{2}g^{n-2}x^2-\&c.}{ng^{n-1}}$ $(=a)$: That is to say, if the Value of the foregoing g be $\sqsubset$ a, the Value of the following g will be also $\sqsubset$ a.

But $\frac{N-g^n}{ng^{n-1}}$ is $\sqsupset$ o; therefore $g+\frac{N-g^n}{ng^{n-1}}$ $(=$ following $g)$ is $\sqsupset$ g: That is to say each following g is less in Value than the foregoing g: But (by what has been abovesaid) still greater than the Value of a:

Whence (since the 2[d] Case is reduced to the 3[d], as it is by one only Renewal) the Approximations in the 2[d] and 3[d] Cases are demonstrated.

* i.e. *The Value of* x

From the foregoing Demonstration it is obvious that as * x, secluding its Sign, is small in respect of g, the Series must converge accord-

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* For $\frac{\frac{1}{n}-2}{3}$, or $\frac{\frac{1}{n}-3}{4}$, or $\frac{\frac{1}{n}-4}{5}$, &c. is = Negative, which $\times$ affirmative $\times -z$ produces an Affirmative; and this multiplied by the next preceding Term, which is Negative will produce a Negative.

ingly swift; for the Difference between g and a being only the said x, that of the next g and a will be only $\frac{n-1}{2} g^{-1} xx + \&c.$ which, if the said x be but very small in respect of g, is but very little more than $\frac{n-1}{2g} xx$. Now this Quantity must be much less than the said x if the said x be much less than g.

This Series converges so swift, not only in the foregoing simple Equations; but also in all affected Equations, when you have got so many of the Figures of the Root you are about extracting, as to free it from the Intanglements and Incumbrances of the other Roots of the Equation, as intirely or almost to double the true Figures in the assumed g by each Renewal.

In the first Edition of this Book I have demonstrated that g being assumed greater than the greatest Value of the Root a in any Equation, the said g would converge to that greatest Value of a provided it was not imaginary. But, because of that Proviso, I will not insert here what I have already done upon this Subject, but leave it to be improved by such as please to do it. That Defect may be remedied in several Cases by particular Methods; as by * changing the Negative Roots into Affirmatives, and the affirmative Roots into Negatives; or by supposing a Fraction whose Numerator is 1, as $\frac{1}{y}$ = the Root a, &c. But all those Methods do not extend the Demonstration to Universality; and withal the Root sought may not be the Greatest, nor any of them that may be reduc'd to such by the above-mentioned Methods: Wherefore

The surest Way of extracting the Roots of affected Equations is to begin the Operation with or by the numeral Exegesis, and so go on with it till the Divisor takes Place, and then pursue it by this Method, or by any of Dr. *Halley*'s Theorems, either of which will serve to find as many Figures of the Root you seek (it being, I presume, not imaginary) as are requisite.

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* See *p.* 113. and 114.

Note, It will, for the most Part, be sufficient to find the first Member of the Root you seek, for to apply this Theorem thereto.

I think it needless to insert here the Universal Theorem from which Mr. *Raphson*'s 2d Set of Theorems are deducible; and therefore will only deduce from the foregoing Universal Theorem some particular ones, by some of which I shall extract some of the Roots of such Equations as I shall chuse for Examples.

$$\text{If}\left\{\begin{array}{c}\text{Equation}\\ aa=N\\ aaa=N\\ a^4=N\\ a^2\pm pa=N\\ a^2-pa=-N\\ a^3\pm pa^2\pm qa=N\\ a^4\pm pa^3=N\\ \text{\&c.}\end{array}\right\}\begin{array}{c}\text{Then by}\\ \text{the}\\ \text{General}\\ \text{Theorem}\end{array}\left\{\begin{array}{c}\frac{N-gg}{2g}\\ \frac{N-g^3}{3gg}\\ \frac{N-g^4}{4g^3}\\ \frac{N-gg\mp pg}{2g\pm p}\\ \frac{-N-gg+pg}{2g-p}\\ \frac{N-g^3\mp pgg\mp qg}{3gg\pm 2pg\pm q}\\ \frac{N-g^4\mp pg^3}{4g^3\pm 3pgg}\\ \text{\&c.}\end{array}\right\}=x.$$

 Example

Example I.

If $aa = 2$; what is a equal to?

Suppose $g = 1$, then $x \left(= \frac{N - gg}{2g} \right) = \frac{2 - 1}{2} = .5$;

Therefore $g + x = 1 + .5 = 1.5 = 2^d\, g$.

Then x (or $2^d\, x$) $= \frac{2 - 2.25}{3} = -.083\ldots$;

Therefore $1.5 - .083 = 1.417 = 3^d\, g$:

Then x (or $3^d\, x$) $= \frac{2 - 2.007889}{2.834} = -.002783\ldots$;

Therefore $1.417 - .002783 = 1.414217 = 4^{th}\, g$:

Then x (or $4^{th}\, x$) $= \frac{2 - 2.000007223089}{2.828434} =$ $-.0000034137622\ldots$;

Therefore $1.414217 - .0000034137622 =$ $1.414213562378 = g$ the $5^{th} = a$ sought nearly. Answer.

Example II.

If $a^4 - 4a^3 = 13824$; 'tis required to find one of the affirmative Values of a.

First I suppose $a = 10$; then $a^4 - 4a^3$ $(= 10000 - 4000)$ $= 6000$; but $6000 \sqsupset 13824$; therefore $a \sqsubset 10$.

Again I suppose $a = 20$; then $a^4 - 4a^3$ $(= 160000 - 32000) = 128000$; but $128000 \sqsubset 13824$; therefore $a \sqsupset 20$; but $\sqsubset 10$, and nearer to 10 than to 20: Therefore suppose $g = 10$

Then $x \left(= \frac{N - g^4 + pg^3}{4g^3 - 3pgg} \right) = \frac{13824 - 10000 + 4000}{4000 - 1200} =$ $\frac{7824}{2800} = 2\ldots$; therefore $g + x = 10 + 2 = 12 = g$ the 2^d $= a$ sought.

Dr. *Edmund Halley* found two other Universal Theorems for Evolving Equations, the one Rational, and the other Irrational, which are better than Mr. *Raphson*'s; for every Renewal of each of Dr, *Halley*'s Theorems Trebles the true Figures in the Value of g, The Method of finding which two Theorems is as follows.

Let

Let us reassume the foregoing Universal Equation; *viz.* $a^n \pm p a^{n-1} \pm q a^{n-2} \pm r a^{n-3} \pm$ &c. $= \pm N$: And suppose $g + x = a$; then the said Equation will become $\overline{g+x}|^n \pm p \times \overline{g+x}|^{n-1} \pm q \times \overline{g+x}|^{n-2}$
$\pm r \times \overline{g+x}|^{n-3} \pm$ &c. $= \pm N$.

That is, by Sir *Isaac Newton*'s Theorem, and Transposition,

$$\left.\begin{array}{l} g^n + n g^{n-1} x + n \times \frac{n-1}{2} g^{n-2} x^2 + \&c. \\ \pm p \times : g^{n-1} + \frac{n-1}{1} g^{n-2} x + \frac{n-1}{1} \times \frac{n-2}{2} g^{n-3} x^2 + \&c: \\ \pm q \times : g^{n-2} + \frac{n-2}{1} g^{n-3} x + \frac{n-2}{1} \times \frac{n-3}{2} g^{n-4} x^2 + \&c: \\ \pm r \times : g^{n-3} + \frac{n-3}{1} g^{n-4} x + \frac{n-3}{1} \times \frac{n-4}{2} g^{n-5} x^2 + \&c: \\ \&c. \\ \mp N. \end{array}\right\} = 0$$

Now in this Equation for the Sum of all its Members, wherein x or any of its Powers is not inserted, substitute $-b$; that is to say, let $-b$ be $= g^n \pm p g^{n-1} \pm q g^{n-2} \pm r g^{n-3}$ $\pm$ &c. $\mp N$.

Also for the Sum of the Cœfficients of the Root x substitute $+c$; that is to say, Let $+c$ be $= n g^{n-1} \pm \overline{pn - p} g^{n-2}$ $\pm \overline{qn - 2q} g^{n-3} \pm \overline{rn - 3r} g^{n-4} \pm$ &c.

Again for the Sum of the Coefficients of xx put $+d$.

Also for the Sum of the Coefficients of x^3, x^4, x^5, &c. put $+f$, $+h$, $+k$, &c. respectively.

Then the foregoing Equation will become $-b + cx$ $+ dxx + fx^3 + hx^4 + kx^5 +$ &c. $= 0$.

Now by rejecting all the Members in this Equation wherein any Power of x exceeding its Square is included, we have $-b + cx + dx^2 = 0$.

And, by Transposition and Division, $x = \frac{b}{c + dx}$:

But

But x, by Mr. *Raphson*'s first Set of Theorems; that is, by our foregoing Universal Theorem is $=\frac{b}{c}$ (for g being supposed near the Value of a, x will be but small; and therefore nearly $=\frac{b}{c}$); consequently $x\left(=\frac{b}{c+dx}\right)=\frac{b}{c+d\times\frac{b}{c}}$ $=\frac{b}{c+\frac{db}{c}}$ which is Dr. *Halley*'s rational Theorem for extracting the Roots of any Equation whatsoever.

Again $-b+cx+dx^2+$ &c. is (by what was before said) $=0$; therefore by rejecting all the Members of this Equation which include any Power of x exceeding xx, and afterwards transposing, we have $dxx+cx=b$.

And by dividing each Part by d, we have

$$xx+\frac{c}{d}x=\frac{b}{d}$$

By Comp. □. $xx+\frac{c}{d}x+\frac{cc}{4dd}=\frac{b}{d}+\frac{cc}{4dd}$ $=\frac{bd+\frac{1}{4}cc}{dd}$

And, by Evolution and Transposition, $x=\frac{\pm\sqrt{:bd+\frac{1}{4}cc:}-\frac{1}{2}c}{d}$

That is to say, if $-\frac{1}{2}c$ be $\sqsubset$ o, then $x=\frac{+\sqrt{:bd+\frac{1}{4}cc:}-\frac{1}{2}c}{d}$;

But if $-\frac{1}{2}c$ be $\sqsupset$ o, then $x=\frac{-\sqrt{bd+\frac{1}{4}cc}-\frac{1}{2}c}{d}$;

which is Dr. *Halley*'s Irrational Theorem for evolving all Equations whatsoever.

As to the Extraction of the Roots of pure Powers by these Theorems, it is manifest, g being taken near the Value of a, and, of consequence, cx, secluding the Sign of the Value of x, considerably greater than the following Term dxx, and this considerably greater than fx^3, &c: that if g be taken $\sqsupset a$, the Value of the next g found by the Rational

Rational Theorem is less, but that found by the Irrational Theorem greater than a: But if g be taken $\sqsubset a$; then the Value of the next g found by the Rational Theorem is greater, but that found by the Irrational Theorem is less than a:

For x, by the Rational Theorem, $= \dfrac{b}{c + \dfrac{bd}{c}} = \dfrac{bc}{cc + bd}$

$=$ (because b is equal to $cx + dx^2 + fx^3 + \&c.$)
$\dfrac{ccx + cdx^2 + cfx^3 + \&c.}{cc + cdx + ddxx + \&c.} = x + \dfrac{cf - dd}{cc} x^3, \&c. =$

(since $\dfrac{nn - n}{2} \times \dfrac{nn - n}{2}$ is* $\sqsubset n \times \dfrac{n^3 - 3nn + 2n}{6}$, and, of consequence, $dd \sqsubset cf$) x — an Affirmative Quantity much less than x, if x be $\sqsubset 0$; but $= x +$, if x be $\sqsupset 0$: consequently what is abovesaid in Relation to the Rational Theorem is true.

And what is abovesaid of the Irrational Theorem will presently appear by only viewing the true Value of x, along with the supposed Value of x exhibited by the Irrational Theorem:

Wherefore these two Theorems are happily coupled, and worthy of that learned and excellent Author.

By a few Examples these two Theorems will be farther explain'd.

Example I.

If $a^3 = 2$, 'tis required to find the Value of a by Dr. *Halley*'s Theorems.

Let $g + x$ be $= a$, and
Suppose $g = 1$; then

$$\left.\begin{array}{l} a^3 = 1 + 3x + 3xx + x^3 \\ -N = -2 \end{array}\right\} = 0$$

$$-1 + 3x + 3xx + x^3 = 0$$

That is $-b + cx + dxx + fx^3 = 0$:

* For the Index n is supposed $\sqsubset 1$.

Then

Then, by Dr. *Halley's* Rational Theorem, $x \left(= \frac{b}{c + \frac{bd}{c}}\right)$

$= \frac{1}{3 + \frac{3}{3}} = \frac{1}{4} = .25.$

And, by his Irrational Theorem, $x \left(= \frac{\sqrt{:bd + \frac{1}{4}cc:} - \frac{1}{2}c}{d}\right)$

$= \frac{\sqrt{:3 + 2\frac{1}{4}:} - \frac{3}{2}}{3} = .26\ldots$

Therefore $g + x$ is, by the Rational Theorem, $= 1.25$; or by the Irrational Theorem $= 1.26$.

Either of which Values of $g + x$ may be taken for g the 2[d]; but the latter is nearest the Value of a; Let us therefore suppose g (or 2[d] g) $= 1.26$: Then

$$\left.\begin{array}{l} a^3 = 2.000376 + 4.7628x + 3.78x^2 + x^3 \\ -N = -2 \end{array}\right\} = 0$$

$$.000376 + 4.7628x + 3.78xx + x^3 = 0$$

That is $-b + cx + dxx + fx^3 = 0$

Then, by the Rational Theorem, $x \left(= \frac{b}{c + \frac{bd}{c}}\right) =$

$\frac{-.000376}{4.7628 - \frac{.000376 \times 3.78}{4.7628}} = -.000078950104920\ldots$; therefore $g + x = 1.26 - .0000789501049 = 1.2599210498950$ $= a$ nearly.

Or, by the Irrational Theorem, $x \left(= \frac{\sqrt{:bd + \frac{1}{4}cc:} - \frac{1}{2}c}{d}\right) =$

$\sqrt{: -\frac{.000376}{3.78} + .3969:} - .63 = \sqrt{.3968005291005291000}$ $\ldots - .63 = .6299210498949\ldots - .63$; therefore $g + x =$ $1.2599210498949 = a$ nearly.

Example II.

Let it be required to find one of the Affirmative Values of a in this adfected Cubick Equation, *Viz.*

$a^3 + 438a^2 - 7825a - 98508430 = 0.$

Let

Let $g + x$ be $= a$; and

1. suppose g equal 300; then

$$\left.\begin{array}{rl} a^3 = & 27000000 + 270000x + 900xx + x^3 \\ +438aa = & 39420000 + 262800x + 438xx \\ -7825a = & -\ 2347500 - \ 7825x \\ -N = & -\ 98508430 \end{array}\right\} = 0$$

$$-34435930 + 524975x + 1338xx + x^3 = 0$$

That is $\quad - \ b \ + \ cx \ + \ dxx + fx^3 = 0$

Then, by the Rational Theorem, $x \left(= \dfrac{b}{c + \dfrac{bd}{c}} \right) = 56 \ldots$

Therefore $g + x = 356$.

Or, by the Irrational Theorem, $x \left(= \dfrac{\overline{bd + \frac{1}{4}cc}|^{\frac{1}{2}} - \frac{1}{2}c}{d} \right)$
$= 57 \ldots$

Therefore $g + x = 357$.

2. Renew the Theorem, and suppose g (or 2d g) $= 357$; Then

$$\left.\begin{array}{rl} a^3 = & 45499293 + 382347x + 1071xx + x^3 \\ +438aa = & 55822662 + 312732x + \ 438xx \\ -7825a = & -\ 2793525 - \ 7825x \\ -N = & -\ 98508430 \end{array}\right\} = 0$$

$$+20000 + 687254x + 1509xx + x^3 = 0$$

That is $- \ b \ + \ cx \ + \ dxx + fx^3 = 0$

Then, by the Rational Theorem, $x \left(= \dfrac{b}{c + \dfrac{bd}{c}} \right)$
$= -.029103118169 \ldots$; therefore $g + x = 357 - .029103118169 = 356.970896881830 = a$ nearly.

And, by the Irrational Theorem $x \left(= \dfrac{\sqrt{bd + \frac{1}{4}cc} - \frac{1}{2}c}{d} \right)$
$= -.029103118180 \ldots$;

Therefore $g + x = 356.9708968181 9 = a$ nearly.

Example III.

Let it be required to find one of the Values of a in this adfected Biquadratick Equation, *viz.* $a^4 - 80a^3 + 1998a^2 - 14937a + 5000 = 0$.

 That

That this Equation may be the easier managed, suppose $\mathrm{a} = \frac{1}{10}a$; then the foregoing Equation will become nearly equal to $\mathrm{a}^4 - 8\,\mathrm{a}^3 + 20\,\mathrm{a}^2 - 15\,\mathrm{a} + .5 = 0$.

And for the first Supposition, let $g + x$ be $= \mathrm{a}$, and $g = 1$; then

$$\left.\begin{array}{rcl} \mathrm{a}^4 & = & +\,1 + 4x + 6xx + 4x^3 + x^4 \\ -\,8\,\mathrm{a}^3 & = & -\,8 - 24x - 24xx - 8x^3 \\ +\,20\,\mathrm{a}^2 & = & +\,20 + 40x + 20xx \\ -\,15\,\mathrm{a} & = & -\,15 - 15x \\ +\,.5 & = & +\,.5 \end{array}\right\} = 0$$

$$-1.5 + 5x + 2xx - 4x^3 + x^4 = 0$$

That is $-\,b + cx + dxx + fx^3 + bx^4 = 0$

Then, by Dr. *Halley*'s Rational Theorem,

$$x\left(= \frac{b}{c + \frac{bd}{c}}\right) = \frac{1.5}{5 + \frac{3}{5}} = .26\ldots$$

Or, by his Irrational Theorem,

$$x\left(= \frac{\overline{bd + \frac{1}{4}cc}|^{\frac{1}{2}} - \frac{1}{2}c}{d}\right) = \frac{\sqrt{:3 + 6\frac{1}{4}:} - 2\frac{1}{2}}{2} = .27\ldots$$

Whence 'tis manifest, that 1.26 or 1.27 is nearly $= \mathrm{a}$; and consequently 12.6 or 12.7 nearly $= a$.

Now let us suppose $g + x = a$, and $g = 12.7$; then

$$\left.\begin{array}{rcl} a^4 & = & +\,26014.4641 + 8193.532x + 967.74x^2 + 50.8x^3 + x^4 \\ -\,80a^3 & = & -\,163870.64 - 38709.6x - 3048x^2 - 80x^3 \\ +\,1998a^2 & = & +\,322257.42 + 50749.2x + 1998x^2 \\ -\,14937a & = & -\,189699.9 - 14937x \\ +\,\mathrm{N} & = & +\,5000 \end{array}\right\} = 0$$

$$\left.\begin{array}{l} -\,298.6559 + 5296.132x - 82.26xx - 29.2x^3 + x^4 \end{array}\right\} = 0$$

$$\left.\begin{array}{l} -\,b + cx - dxx + fx^3 + bx^4 \end{array}\right\} = 0$$

Then, by the Rational Theorem, $x\left(= \frac{b}{c + \frac{bd}{c}}\right) =$

$$\frac{298.6559}{5296.132 - \frac{24567.434334}{5296.132}} = .0564407\ldots;$$ therefore

$g + x = 12.75644$, which is nearly $= a$.

Or, by Dr. *Halley*'s Irrational Theorem,

$$x \left(= \frac{\sqrt{bd + \frac{1}{4}cc} - \frac{1}{2}c}{d} \right) = \frac{\sqrt{6987686.106022} - 2648.066}{-82.26}$$

$= .05644080331\ldots$ which really is less than the true Value of x: But, in order to correct it, substract $\frac{\frac{1}{2}fx^3 + \frac{1}{2}hx^4}{\sqrt{:bd + \frac{1}{4}cc:}}$ $= -\frac{.0026201\ldots}{2643.423\ldots} = -.00000099117\ldots$ from it, and you will have $.05644179448 = x$ corrected.

And, if you desire yet more Figures of the Root; from the x corrected let there be made $-d \times :fx^3 + hx^4: = -.43105602423\ldots$; and $\frac{-\frac{1}{2}c + \sqrt{:bd + \frac{1}{4}cc - dfx^3 - dhx^4:}}{d} =$

$$\frac{-2648.066 + \sqrt{6987685.67496597577\ldots}}{-82.26} =$$

$.05644179448074402\ldots = x$; wherefore $g + x = 12.75644179448074402 = a$ nearly.

The Reason or Demonstration of the foregoing Corrections is this; *Viz.* By what hath been already said and done, any Equation may be reduc'd equal to the following one, *viz.* $-b + cx + dx^2 + fx^3 + hx^4 + \&c. = 0$.

And, by Transposition and Division

$$x^2 + \frac{c}{d}x = \frac{b - fx^3 - hx^4 - \&c.}{d}$$

By Comp. □.

$$xx + \frac{c}{d}x + \frac{cc}{4dd} = \frac{cc}{4dd} + \frac{b - fx^3 - hx^4 - \&c.}{d}$$

By Evolution and Transposition $x =$

$$\frac{-\frac{1}{2}c \pm \sqrt{:\frac{1}{4}cc + bd - dfx^3 - dhx^4 - \&c:}}{d}$$ which is

the last Correction.

And,

And, by Sir *Is. Newton*'s Theorem,

$$\pm\sqrt{:\tfrac{1}{4}cc+bd-dfx^3-dhx^4-\&c:}\ \text{is}=\pm\sqrt{:\tfrac{1}{4}cc+bd:}-\frac{\tfrac{1}{2}dfx^3+\tfrac{1}{2}dhx^4+\&c.}{\pm\sqrt{:\tfrac{1}{4}cc+db:}},\ \&c;\ \text{therefore}$$

$$\frac{-\tfrac{1}{2}c\pm\sqrt{:\tfrac{1}{4}cc+bd-dfx^3-dhx^4-\&c:}}{d}=$$

$$\frac{-\tfrac{1}{2}c\pm\sqrt{:\tfrac{1}{4}cc+bd:}}{d}-\frac{\tfrac{1}{2}fx^3+\tfrac{1}{2}hx^4+\&c.}{\pm\sqrt{:\tfrac{1}{4}cc+bd:}},\ \&c.$$

Now, the Value of x being very small, this last *&c.* may be rejected, as being abundantly less (secluding the Sign of the Value of x) than $\frac{\tfrac{1}{2}fx^3+\tfrac{1}{2}hx^4+\&c.}{\pm\sqrt{:\tfrac{1}{4}cc+bd:}}$; and so you have the first Correction.

CHAP. III.

Of Logarithms.

DEFINITION.

LOgarithms are a Series or Set of artificial Numbers accommodated to an other of natural Numbers, in such sort that the Sum of the Logarithms (or artificial Numbers) of any two (natural) Numbers is equal to the Logarithm of the Product of the two (Natural) Numbers; and consequently the Remainder of the Logarithms of any two Numbers is equal to the Logarithm of the Quotient of these two Numbers; as also two, three, four, *&c.* times, and one half, one third, one fourth, *&c*, The Logarithm of any Number is equal to the Logarithm of the Square, Cube, Biquadrate, *&c.* and of the Square Root, Cube Root, Biquadrate Root, *&c.* of that Number respectively: Or, as the learned Dr. *Halley* defines them, Logarithms are the Indexes of the Ratios of Numbers one to an other.

LEM-

LEMMA.

If any Indefinite Root be extracted out of each of any two given Numbers, and from each of these two Roots an Unite be substracted; I say the Sum of the two indefinitely little Remainders ϕ is equal to the indefinitely little Remainder of the said Indefinite Root of the product of the said two given Numbers, an Unite being from it subducted.

In order to demonstrate this *Lemma*, I will suppose any indefinite Number $= n$; as also one of the two given Numbers (which I suppose are Affirmative) to be $=$ some Binomial or Residual whose first Member is 1, as (suppose) $1 \pm x$; that is to say, if that given Number be $\sqsubset$ 1, then it is $= 1 + x$, but if it be $\sqsupset$ 1, then it is $= 1 - x$: And let the other given Number be $=$ some (finite) Power of $1 \pm x$, as (suppose) $\overline{1 \pm x}|^m$. Then tho' the Value of $1 \pm x$ were determined, that of $\overline{1 \pm x}|^m$ will notwithstanding be $=$ any Number you please, x being not $= 0$, and m being indetermin'd: Thus tho' $1 + x$ were $=$ any given Number that is greater than 1, 'tis manifest $\overline{1 + x}|^m$ shall be $= a$, that is to say $=$ any Affirmative Number whatsoever, if m be $= \frac{L\,a}{L\,1 + x}$: Or $\overline{1 - x}|^m$ is $= a$, if m be $= \frac{L\,a}{L\,1 - x}$.

NB. L *stands for Logarithm.*

Now what we are to demonstrate is, that

$$\overline{1 \pm x}|^{\frac{1}{n}} - 1 + \overline{1 \pm x}|^{\frac{m}{n}} - 1 \;\phi\; \text{is} = \overline{\overline{1 \pm x} \times \overline{1 \pm x}|^m}|^{\frac{1}{n}} - 1 = \overline{1 \pm x}|^{\frac{m+1}{n}} - 1.$$

DEMONSTRATION.

By Sir *Is. Newton*'s Theorem,

1 $$\overline{1 \pm x}|^{\frac{m}{n}} \text{ is} = 1 \pm \frac{m}{n}x + \frac{m}{n} \times \frac{\frac{m}{n} - 1}{2}xx \pm \frac{m}{n} \times \frac{\frac{m}{n} - 1}{2} \times \frac{\frac{m}{n} - 2}{3}x^3 + \frac{m}{n} \times \frac{\frac{m}{n} - 1}{2} \times \frac{\frac{m}{n}\;2}{3} \times \frac{\frac{m}{n} - 3}{4}x^4, \ \&c.$$

But

But since n is indefinite, $\frac{m}{n}$ is indefinitely small, and consequently $\frac{m}{n}$ being added to or taken from any finite Quantity or Number will not sensibly, that is, will only by way of ϕ increase or diminish it; wherefore

2 $\overline{1 \pm x}|^{\frac{m}{n}}$ is $= 1 \pm \frac{m}{n}x + \frac{m}{n} \times \frac{-1}{2} xx\phi \pm \frac{m}{n} \times \frac{-1}{2} \times \frac{-2}{3} x^3 \phi + \frac{m}{n} \times \frac{-1}{2} \times \frac{-2}{3} \times \frac{-3}{4} x^4 \phi$, &c. $= 1 \pm \frac{m}{n}x - \frac{m}{2n}xx \pm \frac{m}{3n}x^3 - \frac{m}{4n}x^4$, &c. ϕ; therefore

3 $\overline{1 \pm x}|^{\frac{m}{n}} - 1 = \pm \frac{m}{n}x - \frac{m}{2n}xx \pm \frac{m}{3n}x^3 - \frac{m}{4n}x^4$, &c. ϕ.

4 From the 3d Step it is manifest that $\overline{1 \pm x}|^{\frac{1}{n}} - 1$ is $= \pm \frac{1}{n}x - \frac{1}{2n}xx \pm \frac{1}{3n}x^3 - \frac{1}{4n}x^4$, &c. ϕ.

5 Also that $\overline{1 \pm x}|^{\frac{m+1}{n}} - 1$ is $= \pm \frac{m+1}{n}x - \frac{m+1}{2n}xx \pm \frac{m+1}{3n}x^3 - \frac{m+1}{4n}x^4$, &c. ϕ, which is equal to the Sum of the 3d and 4th Steps; that is $\overline{1 \pm x}|^{\frac{m+1}{n}} - 1\ \phi = \overline{1 \pm x}|^{\frac{m}{n}} - 1 + \overline{1 \pm x}|^{\frac{1}{n}} - 1$. *Q. E. D.*

SCHOLIUM.

Hence if 1 be substracted from the indefinite-Root of any Number, the Remainder may be taken for the Logarithm of that Number had it not been indefinitely little: But if it be multiplied by 10000 &c. indefinitely (which 10000 &c. indefinitely, in the following Operations, is supposed

posed to consist of as many Figures as n doth) the Product will be finite, and as the said Remainder; that is to say, the Product will be the Logarithm required: As for Example.

RULE I.

If the Logarithm of any Number $= c$; that is, if the Logarithm of the Ratio of 1 to any Number $= c$ be required.

First for the Value of c substitute $1 \pm x$ (that is to say, if c be $\sqsubset 1$, then it is $= 1 + x$, but if it be $\sqsupset 1$, then it is $= 1 - x$):

Then, by what hath been already said, the Logarithm of $1 \pm x$ (or of c) is $=$ 10000 &c. indefinitely $\times : \pm \frac{1}{n}x - \frac{1}{2n}xx \pm \frac{1}{3n}x^3 - \frac{1}{4n}x^4$, &c: $= \beta \times : \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \frac{1}{5}x^5 - \frac{1}{6}x^6$, &c:, putting β

$$= \frac{10000 \text{ \&c. indefinitely}}{n}.$$

By this you may see that Logarithms may be of as many different Forms as the indefinite Number $= n$ can; thus if n be $=$ 10000 &c. indefinitely, then β will be $= 1$; and the Logarithm of $1 \pm x = \pm x - \frac{1}{2}xx \pm \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. which is call'd the Lord *Naper*'s Logarithm of $1 \pm x$: But if n be $=$ 2302585 &c. indefinitely, then β will be $=$ .434294, &c; and the Logarithm of $1 \pm x =$ .434294 &c. $\times : \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \frac{1}{5}x^5$, &c: which is called Mr. *Briggs*'s Logarithm of $1 \pm x$

Again, putting the Terms of any Ratio equal to r and s, of which s is the greater, and $s + r = z$, and $s - r = d$, we have the Logarithm of $\frac{s}{\frac{1}{2}z}$ ($=$ Logarithm of $\frac{s}{\frac{1}{2}s + \frac{1}{2}r} =$ Logarithm of $1 + \frac{\frac{1}{2}s - \frac{1}{2}r}{\frac{1}{2}s + \frac{1}{2}r} =$ Logarithm of $1 + \frac{\frac{1}{2}d}{\frac{1}{2}z}$) $=$

Logarithm

Logarithm of $1+\frac{d}{z}$, which is, by Rule 1. $=\beta\times:\frac{d}{z}-\frac{dd}{2zz}+\frac{d^3}{3z^3}-\frac{d^4}{4z^4}$, &c: And the Logarithm of $\frac{r}{\frac{1}{2}z}=$ Logarithm of $1-\frac{d}{z}=\beta\times-:\frac{d}{z}+\frac{dd}{2zz}+\frac{d^3}{3z^3}+\frac{d^4}{4z^4}$, &c: . Now if from the Logarithm of $\frac{s}{\frac{1}{2}z}$ you substract the Logarithm of $\frac{r}{\frac{1}{2}z}$, you will have

RULE. II.

The Logarithm of $\frac{r}{\frac{1}{2}z})\frac{s}{\frac{1}{2}z}=$ Logarithm of $\frac{s}{r}=\beta\times:\frac{2d}{z}*+\frac{2d^3}{3z^3}*+\frac{2d^5}{5z^5}*+$ &c: $=2\beta\times:\frac{s-r}{s+r}+\frac{1}{3}\times\left.\overline{\frac{s-r}{s+r}}\right|^3+\frac{1}{5}\times\left.\overline{\frac{s-r}{s+r}}\right|^5+$ &c: $=ba+3)ba^3+5)ba^5+7)ba^7+9)ba^9+$ &c, putting $b=2\beta$, and $a=\frac{s-r}{s+r}$ for the Convenience of the Operation. This is an excellent Rule for finding the Logarithms of small Numbers, but, in order to find the Logarithm of a great Number by it, it will be very convenient to consider that great Number as the Product of some small Numbers such whose Logarithms are most of them or all, if you can, already known.

Again, by adding together the Logarithms found, by Rule 1st, of $\frac{s}{\frac{1}{2}z}$ and of $\frac{r}{\frac{1}{2}z}$, we have the Logarithm of $\frac{s}{\frac{1}{2}z}\times\frac{r}{\frac{1}{2}z}$ *viz.* the Logarithm of $\frac{sr}{\frac{1}{4}zz}=\beta\times-:\frac{2dd}{2zz}+\frac{2d^4}{4z^4}+\frac{2d^6}{6z^6}$, &c: And by substracting this Logarithm from the

the Logarithm of 1, that is from o, we have the Logarithm of $\left.\frac{sr}{\frac{1}{4}zz}\right)1 =$ Logarithm of $\frac{\frac{1}{4}zz}{sr} = \beta \times : \frac{2dd}{2zz} + \frac{2d^4}{4z^4} + \frac{2d^6}{6z^6} +$ &c:; And half this Logarithm, *viz.*

RULE III.

$\beta \times : \frac{dd}{2zz} + \frac{d^4}{4z^4} + \frac{d^6}{6z^6} + \frac{d^8}{8z^8} +$ &c : is $=$ the Logarithm of $\frac{\frac{1}{2}z}{\sqrt{rs}}$, *i.e.* of the Ratio of the Geometrical to the Arithmetical Mean between r and s.

Again $\frac{\frac{1}{4}zz}{rs} \left(= \frac{\frac{1}{4}rr + \frac{1}{2}rs + \frac{1}{4}ss}{rs} = \frac{\frac{1}{4}rr - \frac{1}{2}rs + \frac{1}{4}ss + rs}{rs}\right)$ is $= \frac{\frac{1}{4}dd + rs}{rs}$; and its Logarithm is found by (*viz.* substituting $\frac{1}{4}dd + rs$ instead of s, and rs instead of r in) Rule 2, $= 2\beta \times : \frac{\frac{1}{4}dd + rs - rs}{\frac{1}{4}dd + rs + rs} + \frac{1}{3} \times \overline{\frac{\frac{1}{4}dd}{\frac{1}{4}dd + 2rs}}\Big|^3 + \frac{1}{5} \times \overline{\frac{\frac{1}{4}dd}{\frac{1}{4}dd + 2rs}}\Big|^5 +$ &c:; consequently, putting $1 = \frac{1}{4}dd$ and $y = \frac{1}{4}dd + 2rs$, the Logarithm of $\frac{\frac{1}{2}z}{\sqrt{rs}}$ is $= \beta \times : \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} +$ &c: $=$ Logarithm of $\frac{1}{2}z - \frac{Lr + Ls}{2}$ (by the Nature of Logarithms); therefore

RULE IV.

The Logarithm of $\frac{1}{2}z$ is $= \frac{Lr + Ls}{2} + y)\beta + 3y^3)\beta + 5y^5)\beta + 7y^7)\beta +$ &c, which Rule 4 is of excellent

Use for finding the Logarithms of prime Numbers having the Logarithms of the adjoining Numbers given.

Note, *That* $\frac{1}{2}z$ *is always* = *the Number whose Logarithm you seek by Rule* 4: *And since* $\frac{1}{4}dd = 1$, ∵ d *is* = 2, *and consequently* $\frac{1}{2}z + 1 = s$, *and* $\frac{1}{2}z - 1 = r$.

And lastly, let $t = \frac{1}{2}z$ be an odd Number whose Logarithm is sought, and (d being = 2) $t + 1 = s$ and $t - 1 = r$ be equal to even Numbers whose Logarithms are known. Then will the Logarithm of $\frac{s}{r}$ ($= \mathrm{L}\,s - \mathrm{L}\,r$) or of $\frac{t+1}{t-1}$, which, by Rule 2, is $= 2\beta \times : \frac{1}{t} + \frac{1}{3t^3} + \frac{1}{5t^5} + \&c: = u$, be also known. The Logarithm of $\frac{t}{\sqrt{:tt-1:}}$ ($=$ Logarithm of $\frac{\frac{1}{2}z}{\sqrt{sr}}$) is, by the Operation to Rule 4, $= \beta \times : \frac{1}{2tt-1} + \frac{1}{3} \times \overline{\frac{1}{2tt-1}}\Big|^3 + \&c: = \beta \times : \overline{2tt-1}\big|^{-1} + \frac{1}{3} \times \overline{2tt-1}\big|^{-3} + \frac{1}{5} \times \overline{2tt-1}\big|^{-5} + \&c:$. Now this Series, divided by that above expressing the Value of u, yields in the Quotient $\frac{1}{4t} + \frac{1}{24t^3} + \frac{7}{360t^5} + \&c$; consequently this last Series multiplied by $u = u \times : \frac{1}{4t} + \frac{1}{24t^3} + \frac{7}{360t^5} + \&c :$ is $=$ the Logarithm of $\frac{t}{\sqrt{:tt-1:}}$; wherefore

RULE V.

The Logarithm of t is equal $\frac{\mathrm{L}:t+1: + \mathrm{L}:t-1:}{2} + u \times : \frac{1}{4t} + \frac{1}{24t^3} + \frac{7}{360t^5} + \&c:$

This

This 5th Rule is only to be used in finding the Logarithms of great Numbers, or in enlarging a Table of Logarithms: In which case $\frac{L:t+1:+L:t-1:}{2}+\frac{u}{4t}$ is = the Logarithm of t sufficiently near.

Examples.

1. Let it be required to find *Naper*'s Logarithm of 2 or $\frac{2}{1}$ to thirteen Places of Figures.

Here $r=1$, $s=2$, and $\frac{s-r}{s+r} = \frac{1}{3} = a$, and consequently $9)\,1 = aa$. Then, by Rule 2, *Naper*'s Logarithm of 2 will be thus found:

OPERATION.

$2\times\frac{1}{3} = ba = .66666666666666$	$ba = .66666666666666$
$9)ba = ba^3 =$ 7407407407407	$3)ba^3 =$ 2469135802469
$9)ba^3 = ba^5 =$ 823045267489	$5)ba^5 =$ 164609053497
$9)ba^5 = ba^7 =$ 91449474165	$7)ba^7 =$ 13064210595
$9)ba^7 = ba^9 =$ 10161052685	$9)ba^9 =$ 1129005853
$9)ba^9 = ba^{11} =$ 1129005853	$11)ba^{11} =$ 102636895
$9)ba^{11} = ba^{13} =$ 125445094	$13)ba^{13} =$ 9649622
$9)ba^{13} = ba^{15} =$ 13938343	$15)ba^{15} =$ 929222
$9)ba^{15} = ba^{17} =$ 1548704	$17)ba^{17} =$ 91100
$9)ba^{17} = ba^{19} =$ 172078	$19)ba^{19} =$ 9056
$9)ba^{19} = ba^{21} =$ 19119	$21)ba^{21} =$ 910
$9)ba^{21} = ba^{23} =$ 2124	$23)ba^{23} =$ 92
$9)ba^{23} = ba^{25} =$ 236	$25)ba^{25} =$ 9

Naper's Logarithm of 2 = .6931471805599.

2. Let it be required to find *Naper*'s Logarithm of $\frac{5}{4}$ to thirteen places

Here $r=4$, $s=5$, and $\frac{s-r}{s+r} = a = \frac{1}{9}$; consequently $aa = 81)\,1$; then by Rule 2.

 OPE-

OPERATION.

$2\times\frac{1}{9}=ba=$	.2222222222222	$ba=$	.2222222222222
$81)ba=ba^3=$	274348422496	$3)ba^3=$	91449474165
$81)ba^3=ba^5=$	3387017561	$5)ba^5=$	677403512
$81)ba^5=ba^7=$	41815031	$7)ba^7=$	5973575
$81)ba^7=ba^9=$	516234	$9)ba^9=$	57359
$81)ba^9=ba^{11}=$	6373	$11)ba^{11}=$	579
$81)ba^{11}=ba^{13}=$	78	$13)ba^{13}=$	6

Naper's Logarithm of $1\frac{1}{4}=.2231435513142+$

☞ By these two last Examples the Logarithm of 10 may be found; thus

$3\times L2=L2^3=L8=$ 2.0794415416797

$L\frac{5}{4}=$ .2231435513142

Nap. Log. of $8\times\frac{5}{4}=$ *Nap*. Log. of $10=2.302585092994+$

3. Let it be required to find the Value of the Index n for making *Briggs*'s Logarithms, his Logarithm of 10 being 1.

The Question propos'd is the same with this; *viz*;

$\frac{10000 \text{ \&c. indefinitely}}{n}\times$ *Naper*'s Logarithm of 10 is $=1$.

Quere n.

SOLUTION.

$\frac{10000 \text{ \&c. indefinitely}}{n}\times 2.302585092994$ &c. is $=1$ (by the preceeding ☞); consequently (by multiplying each part by n) 10000 &c. indefinitely $\times 2.302585$, &c. $=n=$ 2302585092994 &c. indefinitely $=$ 2302585092994045684 017991454684364207601101488628772976033328 &c. indefinitely as computed by others; therefore

$\frac{10000 \text{ \&c. indefinitely}}{n}=\beta=.43429448190325182765112$ 891891660508229439700580366656611445, &c; and consequently $2\beta=b=.8685889638065036$ &c.

Note,

Note, The Logarithms of 2 and 3 may be expeditiously had by finding the Logarithms of $\frac{3}{2}$ and $\frac{4}{3}$ (by Rule 2), whose Sum is the Logarithm of $\frac{3}{2} \times \frac{4}{3} = 2$; And this Logarithm added to that of $\frac{3}{2}$ is the Logarithm of $\frac{3}{2} \times 2 = 3$: But they along with the Logarithm of 10 may be sooner had by finding the Logarithms of $\frac{16}{15}$, $\frac{25}{24}$ and $\frac{81}{80}$; thus

First for *Naper*'s Logarithm of $\frac{16}{15}$

Here $r = 15$, $s = 16$, and $\frac{16-15}{16+15} = a = \frac{1}{31}$, $\because aa = 961)\,1$. And, by Rule 2.

OPERATION.

$2 \times \frac{1}{31} = ba =$	.0645161290322	$1)ba =$	.0645161290322
$961)ba = ba^3 =$	671343694	$3)ba^3 =$	223781231
$961)ba^3 = ba^5 =$	698588	$5)ba^5 =$	139718-
$961)ba^5 = ba^7 =$	726	$7)ba^7 =$	104-

Naper's Logarithm of $\frac{16}{15}$ = .0645385211375.

Next for *Naper*'s Logarithm of $\frac{25}{24}$

Here $r = 24$, $s = 25$, and $a = \frac{25-24}{25+24} = \frac{1}{49}$; $\because aa = 2401)\,1$; wherefore by Rule 2.

OPERATION.

$2 \times \frac{1}{49} = ba =$	.0408163265306	$ba =$	.0408163265306
$2401)ba = ba^3 =$	169997195	$3)ba^3 =$	56665732-
$2401)ba^3 = ba^5 =$	70802	$5)ba^5 =$	14160
$2401)ba^5 = ba^7 =$	29	$7)ba^7 =$	4

Naper's Logarithm of $\frac{25}{24}$ = .0408219945202

And next for *Naper*'s Logarithm of $\frac{81}{80}$

Here $r = 80$, $s = 81$, and $a = \frac{81-80}{81+80} = \frac{1}{161}$; and $\because aa = 25921)\,1$. Now by Rule 2.

OPE-

OPERATION.

$2\times\frac{1}{161}=ba=.012422360248$4	$ba=.0124223602484$
$25921)ba=ba^3=$ 4792392	$3)ba^3=$ 1597464
$25921)ba^3=ba^5=$ 184	$5)ba^5=$ 37

Naper's Logarithm of $\frac{81}{80}=.0124225199985$

$$
\begin{array}{rcl}
1^{st}=L\frac{16}{15} &=& .0645385211375 \\
2^{d}=L\frac{25}{24} &=& .0408219945202 \\
3^{d}=L\frac{81}{80} &=& .0124225199985 \\
1^{st}+2^{d}+3^{d}=4^{th}=L\frac{9}{8} &=& .1177830356563 \\
1^{st}+2^{d}+4^{th}=5^{th}=L\frac{10}{8} &=& .2231435513140 \\
1^{st}+5^{th}=6^{th}=L\frac{4}{3} &=& .2876820724517 \\
4^{th}+6^{th}=7^{th}=L\frac{3}{2} &=& .4054651081081 \\
6^{th}+7^{th}=8^{th}=L\,2 &=& .6931471805599 \\
7^{th}+8^{th}=9^{th}=L\,3 &=& 1.0986122886681 \\
3\times L\,2=10^{th}=L\,8 &=& 2.0794415416798 \\
5^{th}+10^{th}=11^{th}=L\,10 &=& 2.3025850929940
\end{array}
$$

Note, *Naper*'s Logarithm of 10, and of any other Number being known, *Briggs*'s Logarithm of the ſaid Number may be found by this Proportion.

As *Naper*'s Logarithm of $10=2.30258$ &c.
Is to *Briggs*'s Logarithm of $10=1$
So is *Naper*'s Logarithm of any other Number
To *Briggs*'s Logarithm of the ſaid Number.

But the Index $=b$ being known, the beſt way is to find *Briggs*'s Logarithm a-new: Thus,

Ex. I. Let it be required to find *Briggs*'s Logarithm of 2 to eleven Places.

Note, The Index $=b$ muſt have a Figure, at leaſt, more than the intended Logarithm is to have; therefore in this *Ex.* it muſt have 12 or 13 Figures in it; *viz.* $b=.868588963806$.

The moſt expeditious Method of finding *Briggs*'s Logarithm of 2, that I know, is thus,

$2^{10}=1024$; and, putting $r=1000$ and $s=1024$, we have

have $L\frac{s}{r} = L\,1\frac{24}{1000}$, which (when found by the following Operation) being added to the Logarithm of r which is 3, will give the Logarithm of $\frac{s}{r} \times r$, or of s, *viz.* of 2^{10}; and consequently $\frac{Ls}{10}$ = the Logarithm of 2 sought.

$\frac{s-r}{s+r} = \frac{24}{2024} = \frac{3}{253} = a$; and $\because \frac{9}{64009} = aa$. Now by Rule 2.

OPERATION.

$.86858\&c. \times \frac{3}{253} = ba =$	$.010299473879$	$ba =$	$.010299473879$
$64009)9ba = ba^3 =$	1448159	$3)ba^3 =$	482719
$64009)9ba^3 = ba^5 =$	203	$5)ba^5 =$	$41-$

Briggs's Logarithm $\begin{cases} \text{of } 1\frac{24}{1000} = .010299956639 \\ \text{of } 1000 = 3. \end{cases}$

Briggs's Logarithm $\begin{cases} \text{of } 2^{10} = 3.010299956639 \\ \because \text{of } 2 = .3010299956639 \end{cases}$

Ex. II. Let it be required to find *Briggs*'s Logarithm of 3 to ten or eleven Places.

Since *Briggs*'s Logarithm of 2 is known, his Logarithm of 3 may be soon had by finding his Logarithm of $1\frac{1}{2}$, by *Rule* II, and adding it to the Logarithm of 2, which Sum will give the Logarithm of $1\frac{1}{2} \times 2 = 3$: But it may be sooner found by *Rule* IV. Thus:

$\frac{1}{2}z$ being $= 3$, and $d = 2$, s will be $= 4$, and $r = 2$; and therefore $\frac{1}{4}dd + 2rs = y = 1 + 16 = 17$, and $yy = 289$.

OPERATION.

By *Ex.* I. $\frac{Lr + Ls}{2} = .45154499349$

$17).43429\&c. = y)\beta =$	$2554673423-$
$y)\beta \div 3 \times 289 = 3y^3)\beta =$	2946566
$3y^3)\beta \div \frac{5}{3} \times 289 = 5y^5)\beta =$	$6118-$
$5y^5)\beta \div \frac{7}{5} \times 289 = 7y^7)\beta =$	15

Briggs's Logarithm of $3 = .47712125471$

Ex. III.

Ex. III. Let it be required to find *Briggs*'s Logarithm of 20001, by *Rule* V.

Here $t = 20001$; and *Briggs*'s Logarithm of $\frac{t+1}{t-1} =$ $L\,\frac{20002}{20000} = L\,\frac{10001}{10000} = L\,1.0001$ is supposed to be known, it being $= .0000434272 7687 = u$:

Also $\frac{L:t+1:+L:t-1:}{2} = \frac{L\,20002 + L\,20000}{2} =$

$$\frac{L2+L10001+L2+L10000}{2} = 4.30105170930241 6$$

$$\text{And } \frac{u}{4t} = \underline{\qquad\qquad 542813}$$

Briggs's Logarithm of 20001 $= \underline{4.30105170984523 0}$

Ex. IV. Let it be required to find *Briggs*'s Logarithm of the prime Number 17.

1. Since *Briggs*'s Logarithm of the adjoining Number 16 $(= 2^4)$ which is, by *Ex.* I. $= 4 \times .3010299956639 =$ $1.2041199826559.$, is known, that of 17 is soon found by *Rule* II. from the Logarithm of $\frac{17}{16}$; for this added to the Logarithm of 16 gives the Logarithm of 17.

Here $r = 16$, $s = 17$;

therefore $\frac{17-16}{17+16} = \frac{1}{33} = a$; and $\because$ 1089) $1 = aa$.

OPERATION.

$.868588\&c. \times \frac{1}{33} = ba =$	.0263208776911	$ba =$	.0263208776911
$1089)ba = ba^3 =$	241697683	$3)ba^3 =$	80565894
$1089)ba^3 = ba^5 =$	221944	$5)ba^5 =$	44389
$1089)ba^5 = ba^7 =$	203	$7)ba^7 =$	29

Briggs's Logarithm $\begin{cases} \text{of } \frac{17}{16} = .0263289387223 \\ \text{of } 16 = 1.2041199826559 \end{cases}$

Briggs's Logarithm of $17 = 1.2304489213782$

2, Or,

2. Or, since *Briggs*'s Logarithm of 16, as also of 18 $(= L\,3^2 + L\,2)$ is known, that of 17 may be found by *Rule* IV; thus

$\frac{1}{2}z$ being $= 17$ and $d = 2$, s will be $= 18$ and $r = 16$; and therefore $\frac{1}{4}dd + 2rs = 1 + 576 = 577 = y$; $\therefore yy = 332929$.

OPERATION.

$$\frac{Lr + Ls}{2} = 1.229696243879$$

$$577) \cdot 43429 \text{ \&c.} = y)\beta = \quad 752676745$$

$$y)\beta \div 3 \times 332929 = 3y^3)\beta = \quad 754$$

Briggs's Logarithm of $17 = 1.230448921378$

These two Methods (*viz.* the 2[d] and 4[th] *Rules*) are exact, and expeditious enough in all Cases; and therefore I will not spend any more time in finding such * Methods as are, in some patricular Cases, more expeditious than either of them; those which I mean being not easily reducible to any certain Rules.

SCHOLIUM.

Hence also the Logarithm of any Number being given along with the Index n, the Number itself may be found; thus

$:\overline{1 \pm x}|^{\frac{1}{n}} - 1 : \times 10000$ &c. indefinitely is $=$ Logarithm of $1 \pm x$; wherefore $\overline{1 \pm x}|^{\frac{1}{n}} - 1 = \frac{\text{Logarithm of } 1 \pm x}{10000 \text{ \&c. indefinitely}}$ which suppose $= l$; then $\overline{1 \pm x}|^{\frac{1}{n}} = 1 + l$; and (by equal Involution) $1 \pm x = \overline{1 + l}|^{n} = 1 + nl + \frac{n}{1} \times \frac{n-1}{2} l^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} l^3$, &c. $= 1 + nl + \frac{nn}{2} l^2 + \frac{n^3}{6} l^3 +$

* See *Ex.* I. in Pages 246 and 247.

$\frac{n^4}{24}l^4$, &c. (because n is indefinite).

Note, That as n is indefinitely great, so l is indefinitely small; and therefore nl, $\frac{nn}{2}l^2$, $\frac{nnn}{6}l^3$, &c. are equal to finite Numbers.

But this Series converges very slow especially in great Numbers: Wherefore when the Logarithm of any Number is given along with the Value of the Index n, and the Number itself required, your better way is this; *viz.* seek in the Table of Logarithms the * nearest Logarithm, to that which is given, without their Characteristicks; and the Number answering that and the Characteristick of the given Logarithm call r, if less, or s, if greater than the given Logarithm; and the Number sought, or that whose Logarithm is given, suppose $= u$. And if the Logarithm of s or r in the Table, be not exact enough for your Purpose, you may by the 2[d] or 4[th] preceeding Rule find it to what exactness you please.

Then the Logarithm of u — Logarithm of r, or Logarithm of s — Logarithm of $u =$ Logarithm of $\frac{u}{r}$, or of $\frac{s}{u}$ is (by what hath been said in this Chap.) $= : \overline{\frac{u}{r}}\big|^{\frac{1}{n}} - 1 : \times$ 10000 &c. indefinitely or $= : \overline{\frac{s}{u}}\big|^{\frac{1}{n}} - 1 : \times$ 10000 &c. indefinitely; wherefore

$$\frac{\text{the Logarithm of } \frac{u}{r}}{10000\text{ \&c. indef.}}, \text{ or } \frac{\text{the Logarithm of } \frac{s}{u}}{10000\text{ \&c. indef.}} \text{ is } = \overline{\frac{u}{r}}\Big|^{\frac{1}{n}} - 1,$$

or $= \overline{\frac{s}{u}}\big|^{\frac{1}{n}} - 1 = l$; conseq. $\frac{u}{r}$, or $\frac{s}{u} = \overline{1 + l}\big|^{n}$ is (by Sir

* Thus the nearest such Logarithm to 1.8526627 in a Table of Logarithms for any Number not exceeding 10000 is 3.8526629: And the Number answering 1.8526629 (*viz.* .8526629 and the Characteristick 1 of the given Logarirhm) is $7\frac{123}{100} = 71.23$.

Is.

Is. Newton's Theorem, since n is indefinite) $= 1 + nl + \frac{nn}{2}ll + \frac{n^3}{6}l^3 + \frac{n^4}{24}l^4$, &c. ϕ

Whence $u = r \times : 1 + nl + \frac{1}{2}n^2l^2 + \frac{1}{6}n^3l^3 + \frac{1}{24}n^4 l^4 +$ &c. ϕ : Or u is $= \frac{s}{1 + nl + \frac{1}{2}n^2l^2 + \frac{1}{6}n^3l^3 + \&c. \phi}$

Or since $\overline{1+l}|^n$ is $= \frac{s}{u}$; therefore $\overline{1+l}|^n)\, 1 = \overline{1+l}|^{-n}$ is $= \frac{s}{u})\, 1 = \frac{u}{s}$; that is, by Sir *Isaac Newton*'s Theorem, since n is indefinite, $1 - nl + \frac{1}{2}n^2l^2 - \frac{1}{6}n^3l^3 + \frac{1}{24}n^4l^4 - \frac{1}{120}n^5l^5$, &c. $\phi = \frac{u}{s}$: Whence $u = s \times : 1 - nl + \frac{1}{2}n^2l^2 - \frac{1}{6}n^3l^3$, &c. ϕ :

Again these Values of u may be contracted thus; for Instance, u being $= r + rnl + \frac{1}{2}rn^2l^2 + \frac{1}{6}rn^3l^3 + \frac{1}{24}rn^4l^4 + \frac{1}{120}rn^5l^5 + \frac{1}{720}rn^6l^6 +$ &c. ϕ.

First find a Fractional Quantity whose Numerator (in order to the design'd Contraction) is to consist only of one Member rnl, and the Denominator must have at least two Members, which suppose $= a + e$. Now the two first Members of the Quotient of the Fractional Quantity $\frac{rnl}{a+e}$, to wit $\frac{rnl}{a} - \frac{rnle}{aa}$ must be equal to $rnl + \frac{1}{2}rn^2l^2$ respectively; whence a is $= 1$ and $e = -\frac{1}{2}nl$: Wherefore the Fraction required is $\frac{rnl}{1 - \frac{1}{2}nl}$: And its Quotient is $=$ $rnl + \frac{1}{2}rn^2l^2 + \frac{1}{4}rn^3l^3 + \frac{1}{8}rn^4l^4 + \frac{1}{16}rn^5l^5 + \frac{1}{32}rn^6l^6 +$ &c: From which substract the above Value of u, except its first Member r, and the Remainder is $\frac{1}{12}rn^3l^3 + \frac{1}{12}rn^4l^4 + \frac{13}{240}rn^5l^5 + \frac{43}{1440}rn^6l^6 +$ &c: Whence u is $\sqsupset r + \frac{rnl}{1 - \frac{1}{2}nl}$; that is $u = r + \frac{rnl}{1 - \frac{1}{2}nl} - \frac{1}{12}rn^3l^3 - \frac{1}{12}rn^4l^4 - \frac{13}{240}rn^5l^5 - \frac{43}{1440}rn^6l^6 -$ &c.

Again the Denominator of another Fraction whose Numerator is $\frac{1}{12} r n^3 l^3$ (the first Member of the foregoing Remainder) and the two first Members of whose Quotient are $\frac{1}{12} r n^3 l^3 + \frac{1}{12} r n^4 l^4$ will be found, by the foregoing Method, to be $1 - n l$; And (therefore) the Quotient is $\frac{1}{12} r n^3 l^3 + \frac{1}{12} r n^4 l^4 + \frac{1}{12} r n^5 l^5 + \frac{1}{12} r n^6 l^6 +$ &c; from which substracting the foregoing Remainder there Remains $\frac{7}{240} r n^5 l^5 + \frac{77}{1440} r n^6 l^6 +$ &c: Wherefore the next Fraction ($\frac{7}{240} r n^5 l^5$ being its Numerator, and $\frac{7}{240} r n^5 l^5 + \frac{77}{1440} r n^6 l^6$ the two first Members of its Quotient) will be found (by proceeding as before is taught) $=$

$$\frac{\frac{7}{240} r n^5 l^5}{1 - 1\frac{5}{6} n l}.$$

Whence u is $= r + \frac{r n l}{1 - \frac{1}{2} n l} - \frac{\frac{1}{12} r n^3 l^3}{1 - n l} + \frac{\frac{7}{240} r n^5 l^5}{1 - 1\frac{5}{6} n l}$, &c.

Or u will be found (by a Process like the foregoing one) $= s - \frac{s n l}{1 + \frac{1}{2} n l} + \frac{\frac{1}{12} s n^3 l^3}{1 + n l} - \frac{\frac{7}{240} s n^5 l^5}{1 + 1\frac{5}{6} n l}$, &c. The first Step of which namely $u = r + \frac{r n l}{1 - \frac{1}{2} n l} = r +$

$$\frac{r l \times 10000 \text{ \&c. Indefinitely}}{: \frac{1}{n} - \frac{1}{2} l : \times 10000 \text{ Indefinitely}},$$ or $u = s - \frac{s l}{\frac{1}{n} + \frac{1}{2} l}$ is an Approximation near enough for Practice.

Take one Example *of this* Scholium,

If $a^{365} = 1.06$; the Question is what a is equal to?

The Value of a may, with much Facility, in respect of other Methods, be found by Dr. *Halley*'s Rational Theorem for extracting the Roots of Equations, by supposing $g = 1$; for proceeding by that Theorem with that Value of g, the Value of x will be found $= .0001596$; and consequently $g + x$ ($= 2^d g$) is $= 1.0001596$ which is pretty near equal to the Value of a: But, if it be not near enough for your Purpose, you may renew the Theorem with this Value of g, and the next g will be nearly $= a$. But the Value of a may be sooner had by supposing $g = 1$, and by Dr. *Halley*'s Irrational Theorem, and its Corrections.

I will

I will now shew how to find the Value of a expeditiously and exactly enough by the foregoing *Scholium*; thus:

$a^{365} = 1.06$, by Hypothesis; therefore (by the Nature of Logarithms) $365 \times La = L1.06$ which, by Rule 2. will be found $= .0253058652647$; wherefore $La =$

$$\frac{.02530\text{\&c.}}{365} = .0000693311377 1.$$

Having thus found the Logarithm of a, a it self may be discover'd by the latter Part of our *Scholium* by the help of a Table of Logarithms, thus:

Seek in the Table the nearest Logarithm without its Characteristick, to the above Logarithm of a, and you'll find it to be 4.0000434, whose absolute Number is $1.0001 = r$: But the Logarithm of 1.0001 is (expeditiously found, by Rule 2. to be) $= .0000434272 7686$, which substracted from the above Logarithm of a leaves $.00002590386084$;

Therefore $\dfrac{.00002590386084}{10000\text{\&c. indefinitely}} = l$: And $u = a$ is $= r +$

$$\frac{rl \times 1000\text{\&c. indefinitely}}{:\frac{1}{n} - \frac{1}{2}l : \times 1000\text{ \&c. indefinitely}} = 1.0001 +$$

$$\frac{.00002590645 12}{.434294481 90 - .00001295193} = 1.0001596535874.$$

Before I finish this noble Subject, it will not be improper to insert the following Appendix: But here observe that I suppose the Reader knows how to find by his Table the Logarithm of any Number, whether an Integer or a Fraction of any Sort within the Limits of the Table.

APPENDIX.

1. To find the Logarithm of any Number consisting of one or two Places of Figures more than your Table extends to by the Help of a Table of Logarithms from 1 to at least 10000.

Since the greater any Numbers are in respect of their Differences, the nearer these Differences and the Differences of their Logarithms approach to a Proportionality; therefore

Rule

Rule. Take ſo many of the firſt Figures of the propoſed Number as your Table of Logarithms extends to; and to the Number expreſs'd by theſe Figures add 1: Then place ſo many Cyphers after this Sum, as alſo the ſame Number of Cyphers after the ſaid firſt Figures, that theſe Numbers may be one of them greater, and the other leſs than the propos'd Number. Now the Logarithms of the greater and leſs Numbers may be found by the Table, and from the Conſideration that the Logarithms of $a \times 10$, $a \times 100$ are equal to $La+1$, $La+2$ reſpectively: Wherefore ſay

As the Difference of the greater and leſs Numbers is to the Difference of their Logarithms, ſo is the Difference of the propos'd and leſs Numbers to the Difference of their Logarithms, which added to the Logarithm of the leſs Number gives the Sum = the Logarithm ſought of the propos'd Number nearly.

Examples.

1. Let it be required to find the Logarithm of 123459 by the Help of a Table of Logarithms from 1 to 100000.

The firſt Figures of this Number which this Table extends to are 12345, to which 1 being added, the Sum is 12346: And the propos'd Number is 123459; conſeq.

The greater Number is 123460, whoſe Log. is 5.0915263 = L 12346 + 1

And the leſs Number is 123450 whoſe Log. is 5.0914911 = L 12345 + 1

	10	.0000352
	9	

Wheref. as 10 .. .0000352 :: 9 .. .00003168 = .0000317−

The Logarithm of 123459 is 5.0915228− *Anſ.*

2. Let it be required to find the Logarithm of 1234598 by the help of the abovementioned Table.

The firſt Figures of this Number which this Table extends to are 12345, to which 1 being added the Sum is 12346: And

The

The propos'd Number is 1234598; conseq.
The greater Number is 1234600 whose Log. is 6.0915263
= L 12346 + 2
And the less Number is 1234500 whose Log. is 6.0914911
= L 12345 + 2   100   .0000352
98

Wheref. as 100.. .0000352::98.. .000034496 = .0000345-
The Logarithm of 1234598 is 6.0915256-
Answer.

Note, *The Logarithm - sine of any Number of Degrees, Minutes, Seconds, and, in some Cases, Thirds may be found by the help of the above Rule, and of the common Table of Logarithm - Sines,* &c. *instead of that of the Logarithms; thus*

Suppose it was required to find the Logarithm - sine of $38^\circ\ 40'\ 55''\ 51'''$

The first Figures of this Number, which this Table extends to, are $38^\circ\ 40'$, to which $1'$ being added, the Sum is $38^\circ\ 41'$: And

The propos'd Number is $38^\circ\ 40'\ 55''\ 51'''$; conseq.
The greater Number is 38 41, whose L - S is 9.7958909
And the less Number is 38 40, whose L - S is 9.7957330
0 01   .0001579
0 00 55 51

Now say as 1.. .0001579:: $55''\ 51'''$.. .0001470-

The Logar. - sine of $38^\circ\ 40'\ 55''\ 51'''$ is 9.7958800
Nearly *Answer.*

2. To find the Number that answers to any given Logarithm to one or two Places of Figures more than your Table extends to, by the Help of a Table of Logarithms from 1 to at least 10000.

Rule. When the Logarithm given cannot be exactly found by your Table, take the two nearest Logarithms that are greater and less, as also the Numbers answering those Logarithms, from your Table: Then say,

As

As the Difference of the nearest greater and less Logarithms is to the Difference of the Numbers answering these Logarithms, so is the Difference of the given and less Logarithms to the Difference of the Numbers answering these Logarithms, which, added to the Number answering to the less Logarithm, gives the Number required nearly.

Examples.

1. Let it be required to find the Number answering the Logarithm .4669347 by a Table of Logarithms from 1 to 100000.

The given Logarithm is		.4669347
The nearest that is greater is the Log. of	2.9305 =	.4669417
And the nearest less is the Log. of	2.9304 =	.4669269
	.0001	.0000148
		.0000078

As .0000148 .. .0001 :: .0000078 .. .000053-

The Number required is 2.930453- *Answer.*

2. Let it be required to find the Number of Degrees, Minutes, Sec. and Thirds answering the Log. - sine 9.7958800

The nearest greater is the Log. sine of	38° 41′ =	9.7958909
And the nearest less is the Log. - sine of	38 40 =	9.7957330
	0 01	.0001579
		.0001470

As .0001579 .. 1′ :: .0001470 .. .931′ - = 0 00 55″ 51‴.

The Number of Deg. &c. required is 38.40.55.51 Nearly *Answer.*

PART XVI.

PART XVI.

Of INTEREST.

Interest or the Use paid for the Loan of Money may be either Simple or Compound.

Chap. I. Of Simple INTEREST.

SIMPLE Intereſt is that which is paid for the Loan of any Principal or Sum of Money lent out for ſome time at any Rate *per Cent.* agreed on between the Borrower and the Lender; which formerly according to the Laws of *England*, was 6 Pounds for the Uſe of 100 Pounds for one Year, and 12 Pound for the Uſe of 100 Pound for two Years: And ſo on for a greater or leſſer Sum proportionable to the time propos'd.

There are ſeveral Ways of computing (or anſwering Queſtions about) ſimple Intereſt; as by the ſingle and double Rule of Three. Others make uſe of Tables compoſed at ſeveral Rates *per Cent.* But I ſhall in this Tract ſhew that all Computations relating to ſimple Intereſt are grounded upon Arithmetical Progreſſion; and from thence raiſe ſuch general Theorems as will ſuit with all Caſes. In order to that

Section 1.

Let $\begin{cases} p = \text{any Principal or Sum put to Intereſt.} \\ r = \text{the Ratio of the Rate } per\ Cent.\ per\ Annum. \\ t = \text{the Time of the Principals Continuance at Intereſt} \\ a = \text{the Amount of the Principal and its Intereſt.} \end{cases}$

Note, *the Ratio of the Rate is only the Simple Intereſt of 1 Pound for one Year at any given Rate, and is thus found*

Viz. 100 .. 6 :: 1 .. 0.06 = the Ratio of 6 *per C. p. An.*
Or 100 .. 7 :: 1 .. 0.07 = the Ratio of 7 *per Cent*, &c.
Again 100 .. 7.5 :: 1 .. 0.075 = the Ratio of $7\frac{1}{2}$ *per Cent*, &c.

 And

And if the given Time be whole Years; then $t =$ the Number of those Years: But if the Time given be either pure Parts of a Year, or parts of a Year mix'd with Years, those Parts must be turn'd into Decimals; and then $t =$ those Decimals, &c.

Now the common Parts of a Year may be easily turn'd or converted into Decimal Parts, if it be consider'd

that one $\begin{cases} \text{Day is the } \frac{1}{365} \text{ Part of a Year} = .00274 \text{ fere.} \\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = .0833333 \text{ \&c.} \\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = .25 \\ \text{Half a Year} = .5. \text{ And three Quarters} = .75 \end{cases}$

These Things being premised, we may proceed to raising the Theorems.

Let $r =$ the Interest of 1 Pound for one Year as before.
Then $2r =$ the Interest of 1 Pound for two Years.
And $3r =$ the Interest of 1 Pound for three Years,
$4r =$ the Interest of 1 Pound for four Years,
And so on for any Number of Years propos'd.

Hence it is plain that the simple Interest of 1 Pound for any Time, or number of Years signified by t is $= tr$ Pounds.

Then $\begin{cases} \text{As 1 Pound, is to the Interest of 1 Pound; so is any} \\ \text{Principal or given Sum, to its Interest.} \end{cases}$

That is $1 .. tr :: p .. ptr =$ the Interest of p. Then the Principal being added to its Interest, their Sum will be $= a$ the Amount required: Which gives this general

Theorem $ptr + p = a$

From whence the three following Theorems are easily deduced.

Theorem 2 $\left\{ \frac{a}{tr+1} = p. \right.$ Theorem 3 $\left\{ \frac{a-p}{pt} = r, \right.$

Theorem 4 $\left\{ \frac{a-p}{pr} = t. \right.$

These four Theorems resolve all Questions about simple Interest.

Quest. 1. What will 256 *l.* 10 *s.* amount to in 3 Years, 1 Quarter, 2 Months and 18 Days, at 6 *per Cent. per Annum*?

Here is given $p = 256.5$, $r = 0.06$, and $t = 3.46599$
For 3 Years $= 3$ *Quere a per* Theorem 1.
One Quarter $= 0.25$
2 Months $= 0.16667 = .08333 \times 2$

18 Days $= 0.04932 = .00274 \times 18$

Hence $t = 3.46599$

Then $trp = 3.46599 \times 0.06 \times 256.5 = 53.341586$

And $53.341586 + 256.5 = 309.841586 = trp + p = a$ $= 309$ *l.* 16 *s.* 10 *d.* *Answer.*

Quest. 2. What principal or Sum being put to Interest will raise a Stock of 309 *l.* 16 *s.* 10 *d.* in 3 Years, 1 Quarter, 2 Months and 18 Days; at 6 *per Cent. per An*?

Or the same Question otherwise stated thus:

What is 309 *l.* 16 *s.* 10 *d.* due 3 Years, 1 Quarter, 2 Months and 18 Days hence worth in ready Money, abating or discompting 6 *per Cent. per Annum* simple Interest? Here is given $a = 309.841586$, $r = .06$, $t = 3.46599$ (found as before) thence to find p *per* Theorem 2.

$3.46599 \times .06 + 1 = 1.2079594 = tr + 1$

And $1.2079594)\,309.841586\,(256.5 = p = 256$ *l.* 10 *s.* The Answer required.

Quest. 3. At what Rate of Interest *per Cent*, &c. will 256 *l.* 10 *s.* amount to 309 *l.* 16 *s.* 10 *d.* in 3 Years, 1 Quarter, 2 Months and 18 Days?

Here is given $p = 256.5$, $a = 309.841586$, and $t = 3.46599$; to find r, *per* Theorem 3.

First $309.841586 - 256.5 = 53.341586 = a - p$

Next $3.46599 \times 256.5 = 889.026435 = tp$

And $889.026435)\,53.341586\,(0.06 = r$

Then $1 .. 0.06 :: 100 .. 6$ the Rate required.

Quest. 4. In what time will 256 *l.* 10 *s.* raise a Stock of (or amount to) 309 *l.* 10 *s.* 10 *d.* at 6 *per Cent*, &c?

Here is given $p = 256.5$, $a = 309.841586$ and $r = .06$: To find t *per* Theorem 4.

First $309.841586 - 256.5 = 53.341586 = a - p$

And $256.5 \times .06 = 15.39 = pr$

Then $15.39)\,53.341586\,(3.46599 = t$; that is $t = 3$ Years and 46599 Decimal Parts of a Year, which may be brought into the common Parts of a Year; thus

$.25)\,.46599\,($1 Quarter $.08333)\,.21599\,($2 Months $.00274)\,.04933\,($18 Days.	Hence $t = 3$ Years, 1 Quarter, 2 Months and 18 Days. The Answer required.

It must needs be easy to conceive that what is here done at 6 *per Cent.* may be done at any other Rate of Interest, by forming the Ratio, *viz.* r accordingly.

Sect. 2. *Of Annuities or Pensions in Arrears; computed at simple Interest.*

Annuities or Pensions, &c. are said to be in Arrears, when they are payable or due either Yearly or Half-yearly, &c. and are unpaid for any Number of Payments: Therefore the Business is to compute what all those Payments will amount unto, allowing any Rate of simple Interest for their Forbearance from the Time each particular Payment became due. Now, in order to that

Put $\begin{cases} u = \text{the Annuity, Pension, or yearly Rent, \&c.} \\ t = \text{the Time of its Continuance (or being) unpaid} \\ r = \text{the Ratio, or Interest of 1 } l. \text{ for one Year, as before} \\ a = \text{the Amount of the Annuity and its Interest.} \end{cases}$

Then, If $u =$ the first Years Rent, due without Interest

$ru =$ the Interest $\big\}$ Due at the end of the second
And $2u =$ the Rent $\big\}$ Year

$ru + 2ru =$ the Interest $\big\}$ Due at the end of the third
$3u =$ the Rent $\big\}$ Year

$ru + 2ru + 3ru =$ the Interest $\big\}$ Due at the end of
$4u =$ the Rent $\big\}$ the fourth Year

$ru + 2ru + 3ru + 4ru =$ the Inte. $\big\}$ Due at the end
$5u =$ the Rent $\big\}$ of the fifth Year.

And so on for any Number of Years.

Hence 'tis evident that $ru + 2ru + 3ru + 4ru +$ &c. continued to $t - 1$ Terms $+ tu$ is $= a$; that is (by Part VIII. Chap. I. Step. 17.) $\overline{t-1} \times ru + \frac{\overline{t-1}|^2 \times ru - \overline{t-1} \times ru}{2}$ $\left(= \frac{ttru - tru}{2}\right) + tu = a$. Hence

Theo. 1. $\left\{\frac{tt - t}{2} ru + tu = a.\right.$ Theor. 2. $\frac{2a}{ttr - tr + 2t} = u$

Theo. 3. $\left\{\frac{2a - 2tu}{ttu - tu} = r.\right.$ Theor. 4. $\sqrt{:\frac{2a}{ru} + \frac{xx}{4}:} - \frac{1}{2}x$ $= t$, putting $\frac{2}{r} - 1 = x$.

Quest.

Quest. 1. If 250*l.* yearly Rent (or Pension, &c.) be forborn or unpaid 7 Years; what will it amount to in that time, at 6 *per Cent.* for each Payment as it becomes due?

Here is given $u = 250$, $t = 7$, and $r = .06$ To find a *per* Theorem 1.

First $250 \times 7 = 1750 = ut$; and $1750 \times 7 = 12250 = utt$.

Again $12250 - 1750 = 10500 = ttu - tu$: And $\frac{10500}{2} \times .06 = 315 = \frac{ttur - tur}{2}$.

Lastly $315 + 1750 = 2065 = a$; *viz.* 2065 *l.* is the Answer required.

But if the Annuity Rent or Pension is to be paid by Quarterly or Half-yearly Payments, &c. then $\frac{.06}{2} = .03 = r$ for Half-yearly Payments: And $\frac{.06}{4} = .015 = r$ for Quarterly; or $0.045 = r$ for three Quarterly Payments.

Example of Half-yearly Payments.

Suppose 250*l.* *per Annum* to be paid by Half-yearly Payments were in Arrears or unpaid for 7 Years; What would it amount to, allowing 6 *per Cent. per Annum* for each Payment as it became due?

In this Example there is given $u = 125 = \frac{250}{2}$, $t = 14$ the Number of Payments, and $r = .03 = \frac{0.06}{2}$. Thence to find a

First $125 \times 14 = 1750 = ut$. $1750 \times 14 = 24500 = utt$

Again $24500 - 1750 = 22750 = ttu - tu$

Then $\frac{22750}{2} \times .03 = 341.25 = \frac{ttu - tu}{2} r$

Lastly $341.25 + 1750 = 2091.25 = a$; that is 2091 *l.* 5 *s.* is the Answer required.

N.B. *Hence it may be observed that Half-yearly Payments are more advantageous than yearly.*

For 2091 *l.* 5 *s.* ⊏ 2065 *l.* by 26 *l.* 5 *s.* Consequently Quarterly Payments are more advantageous than Half-yearly Payments.

Quest. 2. What yearly Rent, Pension, &c. being forborn or unpaid 7 Years will raise a Stock of 2065 *l.* allowing 6 *per Cent. per Annum* for each Payment as it becomes due?

Here

Here is given $a = 2065$, $t = 7$, and $r = .06$ to find u *per* Theorem 2.

First $7 \times .06 = .42 = tr$: And $.42 \times 7 = 2.94 = ttr$
Then $ttr - tr = 2.52$: And $ttr - tr + 2t = 16.52$
Lastly $16.52) 4130 (250 = 2a \div : ttr - tr + 2t : = u$
That is 250 *l. per Annum*, &c. will raise 2065 *l.* Stock.

Quest. 3. In what time will 250 *l.* yearly Rent raise a Stock of 2065 *l.* allowing 6 *per Cent.* &c. for the Forbearance of the Payments, as they become due?

Here is given $u = 250$, $a = 2065$, and $r = .06$. To find t *per* Theorem 4.

First $\frac{2}{r} = \frac{2}{.06} = 33.3333$. And $33.3333 - 1 = 32.3333 = x$. Then $16.16666 = \frac{1}{2}x$: And 261.3605 &c. $= \frac{1}{4}xx$

Again $\frac{4130}{15} = 275.3333 = 2a \div ru$. And $275.3333 + 261.3605 = 536.6938 = \frac{2a}{ru} + \frac{xx}{4}$. Then $\surd 536.6938 = 23.16666$

Lastly $23.16666 - 16.16666 = 7 = \surd : \frac{2a}{ru} + \frac{xx}{4} : - \frac{x}{2} = t$: *Viz.* 7 Years is the time required.

Quest. 4. If 250 *l.* yearly Rent, being forborn 7 Years will amount to 2065 *l.* allowing simple Interest for every Payment as it becomes due; What must the Rate of Interest be *per Cent.* &c?

Here are given $u = 250$, $a = 2065$, and $t = 7$. To find r *per* Theorem 3.

$$\text{Thus} \begin{cases} ttu = 12250 \\ -tu = -\underline{1750} \end{cases} \quad \begin{matrix} 4130 = 2a \\ -\underline{3500} = -2tu \end{matrix}$$

$$ttu - tu = 10500) \quad 630 = 2a - 2tu \,(0.06 = r.$$

Then 1 .. 0.06 :: 100 .. 6 the Rate required.

Sect. 3. *The present worth of Annuities, or Pensions, &c. computed at simple Interest.*

The Business of purchasing Annuities, or taking of Leases &c. for any assigned Time depends upon the true Equating of the Principal, or Money laid out on the Purchase, with the Annuity or yearly Rent, by allowing (or discompting) the same Rate of Interest to both: which may be easily perform'd

form'd by duly applying the respective Theorems of the two last Sections together; as will fully appear by the following Question.

Quest. What is 75 *l.* yearly Rent to continue 9 Years worth in ready Money at 6 *per Cent. per Annum* simple Interest?

1. *Per* Theorem 1. of the last *Sect.* find what the propos'd yearly Rent would amount to if it were forborn 9 Years at 6 *per Cent.*

Thus $u = 75$, $t = 9$, and $r = .06$. *Quere* a

$ttu = 6075$ Then 2) 5400 (2700 $\}$ multiply
$tu = \underline{675}$ $\quad$ $\underline{.06}$

$ttu - tu = 5400.$ $\quad$ $162 = : ttur - tur : \div 2$
$\underline{+675 = tu}$
$837 = a$

2. Then by Theorem 2. *Sect.* 1. Find what Principal being put to Interest for the same Time, and at the same Rate, will amount to 837 *l.* $= a$

Thus $tr = 9 \times .06 = .54 \because tr + 1 = 1.54$

And 1.54) 837 (543.5064 $= p$.

That is $p = 543$ *l.* 10 *s.* $1\frac{1}{2}$ *d.* which is the Worth of 75 *l.* a Year, as was required.

From these two Operations (duly consider'd) it must needs be easy to conceive, how the two Theorems by which they were perform'd may be combin'd into one.

For 1. $\dfrac{ttru - tru}{2} + tu = a$. And 2. $ptr + p = a$

Consequently $ptr + p = \dfrac{ttru - tru + 2tu}{2}$. And from this Equation may be deduc'd the following Theorems.

Theorem 1. $\left\{\dfrac{ttr - tr + 2t}{2tr + 2}\right. u = p$. By this Theorem all Questions of the same Kind with the last (*viz.* that above) may be easily and readily answer'd at one Operation.

Theor. 2. $\left\{\dfrac{tr + 1}{ttr - tr + 2t}\right. \times 2p = u$. The. 3. $\left\{\dfrac{2p - 2tu}{ttu - tu - 2pt}\right. = r$

Let $\dfrac{2}{r} - \dfrac{2p}{u} - 1 = \pm x$; then will $tt \pm xt = \dfrac{2p}{ru}$

which gives this Theor. 4. $\left\{\sqrt{:}\dfrac{2p}{ru} + \dfrac{xx}{4} : \mp \dfrac{x}{2} = t.\right.$

By

By the second and fourth Theorems two useful Questions may be easily answer'd.

1. As for Instance: If it be required to find what Annuity, or yearly Rent, &c. may be purchased for any propos'd Sum, to continue any assigned Time, allowing any Rate of Interest.

This Question may be answer'd by Theorem 2.

2. Again: If it be required to find how long any yearly Rent, Pension, or Annuity, &c. may be purchased (or enjoy'd) for any proposed Sum, at any given Rate of Interest.

All Questions of this Kind are easily answer'd *per* Theorem 4.

In these Questions it is suppos'd that the Purchase or yearly Rent is to commence or be immediately enter'd upon. But if it be required to find the Value or Purchase of any Annuity, or yearly Rent, &c. in Reversion; that is, when it is not to be enter'd upon until after some time, or Number of Years are past; then first find what the Sum propos'd to be laid out in the Purchase would amount to, if it were put to Interest during the time the Annuity, &c. is not to be in present Possession; and make that amount the Sum for the Purchase, proceeding with it as in either of the two last Questions, &c.

CHAP. II.

Of *Compound* Interest, and Annuities, *&c.*

COMPOUND Interest is that which arises from any Principal, and its Interest put together as the Interest still becomes due; so that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and, for that Reason, it is called Interest upon Interest, or compound Interest.

As for Instance; suppose 100 *l.* were lent out for two Years at 6 *per Cent. per Annum* compound Interest. Then at the end of the first Year, it will only amount to 106 *l.* as in simple Interest. But for the second Year this 106 *l.* becomes Principal, which will amount to 112 *l.* 7 *s.* 2 $\frac{1}{2}$ *d.* at the second Years End, whereas by simple Interest it would have amounted to but 112 *l.*

And

And altho' it be not lawful to let out Money at compound Intereſt; yet in purchaſing of Annuities or Penſions, &c. and taking Leaſes in Reverſion, it is very uſual to allow compound Intereſt to the Purchaſer for his ready Money; and therefore it is very requiſite to underſtand it.

Sect. 1. *Of Compound Intereſt.*

Let p = the Principal put to Intereſt.
t = the Time of its Continuance. as before.
a = the Amount of the Prin. and Int.
R = the Amount of 1 *l.* and its Int. for 1 Year at any given Rate, which may be thus found.

Viz. 100 .. 106 :: 1 .. 1.06 = the Amount of 1 *l.* at 6 *per cent.*
Or 100 .. 107 :: 1 .. 1.07 = the Amount of 1 *l.* at 7 *per cent.*
And ſo on for any other aſſigned Rate of Intereſt.

Then if R = the Amount of 1 *l.* for one Year at any Rate.
RR = the Amount of 1 *l.* for two Years.
RRR = the Amount of 1 *l.* for three Years.
R^4 = the Amount of 1 *l.* for four Years.
R^5 = the Amount of 1 *l.* for five Years. Here $t=5$.
For $1 .. R :: R .. RR :: RR .. R^3 :: R^3 .. R^4 :: R^4 .. R^5$, &c. in $\div\div$

That is As 1 *l*, is to the Amount of 1 *l.* at one Years end; ſo is that Amount, to the Amount of 1 *l.* at two Years end, &c.

Whence it is plain that compound Intereſt is grounded upon a Series of Terms increaſing in geometrical Proportion continued; wherein t (*viz.* the Number of Years) does always aſſign the Index of the laſt and higheſt Term, *viz.* the Power of R, which is R^t.

Again $1 .. R^t :: p .. pR^t = a$ the Amount of p for the Time that R^t = the Amount of 1 *l.*

That is As 1 *l*, is to the Amount of 1 *l.* for any given Time; ſo is any propos'd Principal (or Sum), to its amount for the ſame Time.

From the Premiſes the Reaſon of the following Theorems may be very eaſily underſtood.

Theorem 1. $pR^t = a$.

From hence the two following Theorems are eaſily deduced.

Theorem 2. $\frac{a}{R^t} = p$. Theorem 3. $\frac{a}{p} = R^t$.

By thefe three Theorems all Queftions about compound Intereft may be truly refolved by the Pen only, *viz.* without Tables, tho' not fo readily as by the help of Tables calculated on purpofe.

But here I will fhew how to folve fuch Queftions by the help of a Table of Logarithms: thus

$pR^t = a$, as above; therefore, by the Nature of Logarithms.

Theorem 1. $Lp + t \times LR = La$

Theorem 2. $La - t \times LR = Lp$

Theorem 3. $\dfrac{La - Lp}{t} = LR$

Theorem 4. $\dfrac{La - Lp}{LR} = t$.

Queft. 1. What will 375 *l.* 10 *s.* amount to in 9 Years at 6 *per cent. per annum* compound Intereft?

Here is given $p = 375.5$, $t = 9$, and $R = 1.06$. To find a *per* Theorem 1.

The Logarithm of 375.5 is 2.5746099
9 × Logarithm of 1.06 is = .2277531
2.8023630 $= Lp + t \times LR = La$;

Therefore a is $= 634.4$ fere $= 634$ *l.* 8 *s.* The Anfwer required.

Queft. 2. What Principal (or Sum) muft be put to Intereft to raife a Stock of 634 *l.* 8 *s.* in 9 Years at 6 *per cent. per annum*, &c?

Here is given $a = 634.4$, $R = 1.06$, and $t = 9$. To find p *per* Theorem 2. Thus

The Logarithm of 634.4 is 2.8023632
9 × Logarithm of 1.06 is = .2277531
2.5746101 $= La - t \times LR = Lp$;

Confequently p is $= 375.5+$; that is $p = 375$ *l.* 10 *s.* which is the Principal (or Sum) as was required.

Queft. 3. In what time will 375 *l* 10 *s.* raife a Stock of (or amount to) 634 *l.* 8 *s.* allowing 6 *per cent. per annum* compound Intereft?

Here is given $a = 634.4$, $p = 375.5$, and $R = 1.06$. To find t by Theorem 4.

The

The Logarithm of 634.4 is 2.8023632
The Logarithm of 375.5 is 2.5746099

$$.0253059)\ .2277533(9 + = \frac{La - Lp}{LR} = t;$$

that is t is $= 9$ the Number of Years required.

Quest. 4. If 375 *l.* 10 *s.* will amount to (or raise a Stock of) 634 *l.* 8 *s.* in 9 Years; what must the Rate of Interest be *per cent. per annum* compound Interest?

Here is given $a = 634.4$, $p = 375.5$, and $t = 9$. *Quere* R *per* Theorem 3.

The Log. of 634.4 is 2.8023632
The Log. of 375.5 is 2.5746099

$$9)\ .2277533(.0253059 = \frac{La - Lp}{t} = LR$$

And the natural Number which answers the Logarithm .0253059 in the Table is 1.06;

Then 1 .. 1.06 — 1 :: 100 . 6 the Rate *per cent* required.

Note, *If the Logarithm of the given Number; or, if the required Number of the given Logarithm, be not exact enough for your Purpose in your Table of Logarithms, you may find them sufficiently exact by the Appendix to* Chap. III. Part XV. *or make them as exact as you please by the said Chapter.*

Sect. 2. *Of Annuities or Pensions in Arrear computed at Compound Interest.*

When Annuities, &c. are said to be in Arrear, see P. 260.

And I shall here make use of the same Letters to represent the same Things as before in that Page, save only that R is here $=$ the Amount of 1 *l.* for 1 Year, as in § 1. of this Chap.

Suppose $u =$ the first Years Rent of any Annuity without Interest

Then will $Ru + u =$ { The Amount of the first Years Rent and its Interest; more the second Years Rent, &c.

And $RRu + Ru + u =$ { The Amount of the 1st and 2d Years Rents with their Interest, more the 3d Years Rent, &c.

Here $RRu + Ru + u = a$ the Amount of any Yearly Rent, or Annuity being forborn 3 Years. And from hence may be deduc'd these Proportions.

Viz. $u \,..\, Ru :: Ru \,..\, RRu :: R^2 u \,..\, R^3 u$, and so on in $\div\!\div$ for any Number of Terms, or Years, denoted by t, wherein the last Term will always be uR^{t-1}; wherefore, by Part VIII. Chap. II. Step 15.

Theor. 1. $a = \frac{:R^t - 1: \times u}{R-1}$; conseq. $L{:}R^t - 1{:} + Lu - L{:}R - 1{:} = La$

Theor. 2. $u = \frac{:R-1: \times a}{R^t - 1}$: conseq. $L{:}R - 1{:} + La - L{:}R^t - 1{:} = Lu$.

Theor. 3. $R^t = \frac{Ra + u - a}{u}$: conseq. $\frac{L{:}Ra + u - a{:} - Lu}{LR} = t$.

Theor. 4. $R^t - \frac{a}{u} R + \frac{a}{u} - 1 = 0$. If this Equation be resolved into Numbers, one of its Roots will shew the Value of R.

Quest. 1. If 30*l.* Yearly Rent, or Annuity, &c. be forborn (*viz.* remain unpaid) 9 Years; what will it amount to at 6 *per cent. per annum* compound Interest?

Here is given $u = 30$, $t = 9$, and $R = 1.06$; to find a *per* Theorem 1.

$LR^t = t \times LR = 9 \times L1.06 = 9 \times .0253059 = .2277531$:

And the *Number answering to this Log. is $1.689480 = R^t$;

Conseq. $R^t - 1 = .68948$, whose Log. is ...... $\bar{1}.8385217$

$Lu = L\,30 = 1.4771213$

Sum 1.3156430

$L{:}R - 1{:} = L.06 = \bar{2}.7781513$

$La = 2.5374917$:

And the Number answering to this Log. of a is $344.74 = a = 344$*l.* 14*s.* $9\frac{1}{2}$*d.* The Amount required.

Quest. 2. What Yearly Rent, or Annuity, &c. being forborn, or unpaid 9 Years, will raise a Stock of 344*l.* 14*s.* $9\frac{1}{2}$*d.* $= 344.74$ *l.* at 6 *per Cent*, &c?

Here is given $a = 344.74$, $t = 9$, and $R = 1.06$; to find u by Theorem 2.

* See the Appendix to the Logarithms; *viz.* to Part XV. Chap. III.

L : R

$$L:R-1:=L.06=\bar{2}.7781513$$
$$La=L\,344.74=2.5374917$$

Sum 1.3156430

$L:R^t-1:$ is (as before found) $=$ $\bar{1}.8385217$

$Lu=1.4771213:$

Consequently $u=30=30\,l.$ *Answer.*

Quest. 3. In what Time will 30 *l.* Yearly Rent raise a Stock of, or amount to 344 *l.* 14 *s.* $9\frac{1}{2}$ *d.* allowing 6 *per Cent. per Annum* for the forbearance of the Payments, &c?

Here is given $u=30$, $a=344.74$, and $R=1.06$; to find t *per* Theorem 3.

$Ra-a+u=365.4244-344.74+30=50.6844$

And the * Logarithm of 50.6844 is 1.7048743

$Lu=L\,30=\underline{1.4771212}+$

$LR=L\,1.06=.0253059$, And $.0253059).2277531$ $(9=t$; that is 9 Years is the Time required.

Quest. 4. If 30 *l. per Annum* being unpaid 9 Years will amount to 344 *l.* 14 *s.* $9\frac{1}{2}$ *d.* allowing compound Interest for every Payment as it becomes due; what must the Rate of Interest be *per cent. per Annum*?

Here is given $u=30$, $a=344.74$, and $t=9$; to find R by Theorem 4.

$\frac{u}{u}=11.491333+$, and $\frac{a}{u}-1=10.491333+$: Hence there is this Equation $R^9-11.491333R+10.491333=0$. The Root sought of, and in, this Equation will be found, by Extraction to be 1.06; but not so easily as may be expected at first Sight: For if *g be suppos'd $=1$, x will be found $=0$, by either *Halley*'s or *Raphson*'s Theorems; because one of the Roots of that Equation is 1; wherefore you must suppose $g=1.05$, 1.06, or 1.07, &c; *viz.* so as to carry the Series beyond the Limits of the Root 1; &c.

☞ *See the* 4th *Question of the next following Sect.*

And $1\,..\,1.06-1::100\,..\,6$ the Rate *per cent. per Annum* required.

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* See the Converging Series.

Sect. 3. *To find the present Worth of Annuities, Pensions or Leases,* &c. *at compound Interest.*

Let p = the present Worth of any Annuity, or Lease, &c. and the rest of the Letters as before.

Then, from what has been said in § 3. Chap. I. about purchasing of Annuities at simple Interest, it will be easy to form the like Theorems here at compound Interest, *viz.* by combining Theorem 1. Page 268, and Theor. 1. Page 265. into one Theorem.

For $\frac{uR^t - u}{R - 1} = a$ { The Amount of any Yearly Rent being unpaid any Number of Years, *per* Theorem 1. of the last §.

And $pR^t = a$ { The Amount of any Principal, or Sum being put to Interest for the same Number of Years, *per* Theorem 1. of § 1.

Hence it follows that $pR^t = \frac{uR^t - u}{R - 1}$; *viz.* $pR^{t+1} - pR^t = uR^t - u$.

Or this Equation may be rais'd from the consideration that purchasing Annuities, or taking of Leases, &c. is grounded upon a Rank or Series of geometrical Proportionals continually decreasing;

Thus $\frac{u}{R}$ is the first and greatest Term, R the common Ratio of all the Terms, and p is the Sum of all the Series: That is $\frac{u}{R} \cdot\cdot \frac{u}{RR} :: \frac{u}{RR} \cdot\cdot \frac{u}{R^3} :: \frac{u}{R^3} \cdot\cdot \frac{u}{R^4} :: \frac{u}{R^4} \cdot\cdot \frac{u}{R^5}$, &c. in $\because$ until the last Term $= \frac{u}{R^t}$. Then (by Part VIII. Chap. II. Step. 15.) will p be $= \frac{\frac{u}{R} \times R - \frac{u}{R^t}}{R - 1}$: consequently $pR^{t+1} - pR^t = uR^t - u$ as above.

From this Equation may be deduced these following Theorems.

Theorem

Theor. 1. $p = \frac{:1 - \frac{1}{R^t}: \times u}{R - 1}$; conseq. $L:1 - \frac{1}{R^t}: + Lu - L:R - 1: = Lp$

Theor. 2. $u = \frac{:R - 1 : \times p}{1 - \frac{1}{R^t}}$; conseq. $L:R - 1: + Lp - L:1 - \frac{1}{R^t}: = Lu$

Theor. 3. $R^t = \frac{u}{p + u - pR}$; conseq. $\frac{Lu - L:p + u - pR:}{LR} = t$

Theor. 4. $R^{t+1} - \frac{u}{p}R^t - R^t + \frac{u}{p} = 0$. The resolving of this Equation will discover the Value of R.

Quest. 1. What is 30 *l.* Yearly Rent to continue 7 Years Worth in ready Money allowing *6 per Cent. per Annum* compound Intereſt to the Purchaſer?

Here is given $u = 30$, $t = 7$, and $R = 1.06$; to find p *per* Theorem 1.

$LR^t = t \times LR = 7 \times L1.06 = 7 \times .0253059 = .1771413$; conſeq. the Log. of $\frac{1}{R^t} = 0 - .1771413 = \bar{1}.8228587$: And the * Number anſwering to this Log. is .6650568; therefore $1 - \frac{1}{R^t} = .3349432$ whoſe Log. is $\bar{1}.5249711$

$Lu = L30 = 1.4771213$

Sum 1.0020924

$L:R - 1: = L.06 = \bar{2}.7781513$

$Lp = 2.2239411$: conſeq.

$p = 167.4716 = 167$ *l.* 09 *s.* 5 *d.* being the Anſwer required.

Quest. 2. What Annuity or Yearly Rent to continue 7 Years may be purchaſed for 167 *l.* 09 *s.* 5 *d.* allowing *6 per Cent. per Annum* compound Intereſt to the Purchaſer?

* See Appendix to Logarithms.

In

In this Queſtion there is given $p=167.4716$, $t=7$, and $R=1.06$; to find u by the 2d Theorem.

$$L:R-1:=L.06=\bar{2}.7781513$$
$$Lp=L\,167.4716=2.2239411$$

Sum 1.0020924

$$L:1-\frac{1}{R^t}: \text{ is (as before found) } =\bar{1}.5249711$$

$Lu=1.4771213$: conſeq. $u=30$; that is 30 *l.* is the Anſwer required.

Queſt. 3. How long may One have a Leaſe of 30 *l.* Yearly Rent for 167 *l.* 09 *s.* 5 *d.* allowing 6 *per Cent. per Annum*, compound Intereſt to the Purchaſer? —— Here is given $p=167.4716$, $u=30$, and $R=1.06$; to find t by the 3d Theor.

First $Lu=L\,30=$ 1.4771213-

2dly $p+u=197.4716$

And $pR=177.5199$-

$p+u-pR=$ 19.9517+and its Log. is 1.2999799+

$LR=L1.06=.0253059$. And .0253059).1771413 ($7=t$; that is ſeven Years is the Anſwer required.

Queſt. 4. Suppoſe one ſhould give 220 *l.* for the Purchaſe of an Annuity of 20 *l.* *per Annum* to continue 21 Years. At what Rate of Intereſt *per cent. per Annum* would the Purchaſe be made, allowing compound Intereſt to the Purchaſer?

In this Queſtion there is given $p=220$, $u=20$, and $t=21$; To find R by Theorem 4.

$\frac{u}{p}=\frac{20}{220}=\frac{1}{11}$; whence, *per* Theorem 4, you have this Equation $R^{22}-1\frac{1}{11}R^{21}+\frac{1}{11}=0$; or $11R^{22}-12R^{21}+1=0$.

Tho' this Equation is of a high Degree, yet the ſought Root therein being but by a ſmall Matter more than 1, one may think, at firſt Sight, that the ſaid Root may eaſily be extracted by ſuppoſing * $g=1$, and by Dr. Halley's Irrational Theorem and its Corrections: But if g be ſuppos'd $=1$, then x will be found $=0$; becauſe 1 is one of the Roots of that Equation: And tho' g be ſuppos'd ſome what more than

* *See the Converging Series.*

than 1; yet the Root 1 does so retard the Series from converging to the sought Root, that this cannot be by that Theorem readily extracted: And, if in order to get rid of the Root 1, the Equation were divided by $R - 1\ (=0)$, the Quotient would consist of a great many Terms; so that I think it much more eligible not to depress the Equation lower by such Division; but reduce it to another Form, &c. thus

Suppose $\frac{1}{y} = R$; and the foregoing Equation will become $\frac{11}{y^{22}} - \frac{12}{y^{21}} + 1 = 0$; consequently $11 - 12y + y^{22} = 0$.

Now Suppose $g + x = y$; then

$$\left.\begin{array}{lll} y^{22} & = & g^{22} + 22g^{21}x + 242g^{20}xx, \text{ \&c.} \\ -12y & = & -12g \quad -12x \\ +11 & = & +11 \end{array}\right\} = 0.$$

If g be suppos'd $= .9$, the Value of x, which will be found, as well by Dr. *Halley*'s Rational, as by his irrational Theorem, is only .034+, which is too small, the true Value of x being then .036+.

Suppose therefore $g = .936$. Then, by the Logarithms the Log. of .936 (=* L 3 . 9 + L . 24) = $\bar{1}$. 971275848 &c.

$22 \text{ L}g = \bar{1} . 3680687$. Number . 2333827

$\text{L } 22 = 1 . 3424227$
$21 \text{ L}g = \bar{1} . 3967928$

Sum . 7392155 Number 5 . 485491

$\text{L } 242 = 2 . 3838154$
$20 \text{ L}g = \bar{1} . 4255170$

Sum 1 . 8093324 Number 64 . 466

* See *Sherwin*'s Mathematical Tables Page 28. or, &c.

 There-

Therefore

$$\left.\begin{array}{lll} y^{22} = & 0.2333827 + 5.485491x + 64.466xx, \text{\&c.} \\ -12y = & -11.232 \quad - 12x \\ +11 = & +11 \end{array}\right\} = 0$$

$$+ .0013827 - 6.514508x + 64.466xx, \text{\&c.} = 0$$

$$- \quad b \quad + cx \quad + \quad dxx, \text{\&c.} = 0$$

And, by Dr. *Halley*'s Rational, or Irrational Theorem, $x = .0002126+$; therefore $g + x = .9362126 = y$ nearly; Consequently $R = 1.068133+$.

And then it will be $1 .. 1.068133 - 1 :: 100 .. 6.8133$ the Rate *per Cent.* &c. as was required.

These four Questions include all the Variety that can be propos'd about purchasing Annuities, or Leases, &c. which are either immediately to be enter'd upon, or in Possession at the Time when the Purchase is made.

But Theorems for solving such Questions as relate to Annuities, or taking Leases, &c. in Reversion may be raised in the following Manner.

1. Suppose it was required to compute the present Worth of 75 *l.* Yearly Rent which is not to commence or be enter'd upon until 10 Years hence, and then to continue 7 Years after that Time, at 6 *per cent.* &c. compound Interest?

Here $u = 75$, $t = 7$, $R = 1.06$; and suppose $T = 10$ the Number of Years the Annuity is not to commence, or be enter'd upon: Then

By the 1st Theorem of this § 3. the present Worth of u Pounds Yearly Rent to continue $T + t$ Years at $R - 1$ *per* 1. &c. will be found $= \dfrac{u - \dfrac{u}{R^{T+t}}}{R - 1}$

And by the said Theorem, the present Worth of u Pounds Yearly Rent to continue T Years at the same Rate of Interest will be also found $= \dfrac{u - \dfrac{u}{R^{T}}}{R - 1}$

* This in Conclusion, is the same with Mr. Ward's *Method.* ☞ See Page 264.

* Conseq. the present Worth required, *viz.*

$$p \text{ is} = \frac{u - \dfrac{u}{R^{T+t}}}{R - 1} - \frac{u - \dfrac{u}{R^{T}}}{R - 1} = \frac{1 - \dfrac{1}{R^{t}}}{R^{T} \times :R - 1:} u.$$

Theorem. 1.

Where-

Wherefore $L:1-\frac{1}{R^t}:+Lu-T\times LR-L:R-1:=Lp.$

By this Equation the preceeding Queſtion may be ſolv'd; thus

$LR^t=t\times LR=7\times L\,1.06=7\times .0253059=.1771410;$

$\therefore L\frac{1}{R^t}=L\,1-t\,LR=\quad \bar{1}.8228590$

And the Number anſwering to this Log. is $.6650572$;

$$\therefore L:1-\frac{1}{R^t}:=L.3349428=\bar{1}.5249706$$

$$Lu=L\,75=1.8750613$$

$$\text{Sum}\quad 1.4000319$$

$$T\times LR=10\times .02530586=\quad .2530586$$

$$L:R-1:=L.06=\bar{2}.7781512$$

$$\text{Sum}\quad \bar{1}.0312098$$

$$2.3688221=$$

Lp: Conſequently $p=233.788=233\,l.$ $15\,s.$ $9\,d.$ the preſent Worth of $75\,l.$ *per Annum* in Reverſion, &c. as was required.

Ex. 2. What Annuity or Yearly Rent to be enter'd upon 10 Years hence, and then to continue 7 Years may be purchaſed for $233\,l.$ $15\,s.$ $9\,d.$ ready Money at 6 *per Cent.* &c. compound Intereſt?

The Theorem for ſolving this Queſtion deducible from the foregoing Theorem 1. is $\frac{pR^T\times:R-1:}{1-\frac{1}{R^t}}=u.$ Theor. 2.

$$\text{Conſeq. } Lp+T\times LR+L:R-1:-L:1-\frac{1}{R^t}:=Lu$$

$$Lp=L\,233.788=2.3688221$$

$$T\times LR=\quad .2530586$$

$$L:R-1:=\bar{2}.7781512$$

$$\text{Sum}\quad 1.4000319$$

$$L:1-\frac{1}{R^t}:=\bar{1}.5249706$$

$$Lu=1.8750613\therefore u=75;$$ that is, $75\,l.$ is the Yearly Rent required by the Queſtion.

These

These two Examples of finding p and u do fully shew the Method to be used in resolving the two general, and indeed most useful Questions about Annuities or Leases in Reversion: And, if there be Occasion, either the Rate or the Time, *viz.* R, t or T may be found by a due Application of their respective Theorems.

Note, *If the Rents or Annuities, &c. are to be paid Half-Yearly, or Quarterly, that*

Then $R = \sqrt{1.06}$ for Half-Yearly } Payments at 6 *per*
And $R = \sqrt[4]{1.06}$ for Quarterly } *Cent.* &c.
&c.

Sect. 4. *Of purchasing Freehold or real Estates at Compound Interest.*

All Freehold or Real Estates are suppos'd to be purchas'd or bought to continue for ever (*viz.* without any limited Time); therefore the Business of computing the true Value of such Estates is grounded upon a Rank or Series of Geometrical Proportionals continually decreasing *ad infinitum*:

Thus let p, u, R denote the same Data as in the last Section;

Then the Series will be $\frac{u}{R}$, $\frac{u}{R^2}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, &c. and so on in $\div\!\div$ until the last Term $=0$. Then will $\frac{\frac{u}{R} \times R - 0}{R - 1}$ $= p$ (by what has been prov'd in $\div\!\div$) $= \frac{u}{R - 1}$.

This Equation affords these following Theorems

Theorem 1. $p = \frac{u}{R - 1}$, *ut supra.* Theor. 2. $pR - p = u$

Theorem 3. $\frac{p + u}{p} = R$

Example, suppose a Freehold Estate of 75 *l.* Yearly Rent were to be sold; what is it worth, allowing the Buyer 6 *per Cent.* &c. compound Interest for his Money?

In this Question there is given $u = 75$, $R = 1.06$; to find p *per* Theorem 1. Thus

R — 1

$R - 1 = .06$) $75 = u$ ($1250\,l. = p$ the Anſwer required.

And ſo for any of the Reſt as Occaſion requires.

But, if the Rent is to be paid either by Half-yearly or Quarterly Payments;

$$\left.\begin{array}{l}\text{Then } R = \sqrt{1.06} \text{ for Half-yearly}\\ \text{And } R = \sqrt{\sqrt{1.06}} \text{ for Quarterly}\end{array}\right\} \begin{array}{l}\text{Payments at 6 \textit{per}}\\ \text{\textit{Cent.} \&c.}\end{array}$$

$$\text{Or}\left\{\begin{array}{l}R = 1.08 \text{ for Yearly}\\ R = \sqrt{1.08} \text{ for Half-yearly}\\ R = \sqrt{\sqrt{1.08}} \text{ for Quarterly}\end{array}\right\} \begin{array}{l}\text{Payments at 8 \textit{per}}\\ \text{\textit{Cent. per Annum.}}\end{array}$$

The like is to be underſtood for any other propos'd Rate of Intereſt either greater or leſs than 6 *per Cent.* &c.

The Application of theſe Theorems to Practice is ſo very eaſy, that it is needleſs to inſert any more Examples.

PART

PART XVII.

Some of Diophantus's *Questions.*

IT may be expected that 'tis possible to solve the following Question if propos'd in more universal Terms; *viz.* that 'tis possible to divide any given Number into two Rational Squares: But that it is not Mr. *Thomas Wallis* of the City of *Corke* has demonstrated; thus

Lemma 1. All even square Numbers are divisible [*viz.* divisible without leaving any Remainder] by 4.

Demonstration. The Roots of all even Squares are even; therefore divisible by 2. Let * $2n$ be = the Root of any even Square. The Square of $2n$ is $4nn$; wherefore all even Squares are divisible by 4.

* NB. *All the Symbols or Letters in these Lemma's, and in the followingConsect. and Scholium are suppos'd to be equal to integer Numbers.*

Lemma 2. Any odd Square Number divided by 4 leaves a Remainder of 1.

Demonstration. The Roots of all odd Square Numbers are odd. Let $2n+1$ be = the Root of any odd Square. The Square of $2n+1 = 4nn+4n+1$; therefore any odd Square divided by 4 leaves 1. [for $4nn+4n+1$ divided by 4 gives $nn+n$ for the Quotient, and 1 for the Remainder.]

Lemma 3. If a Number consist of two even Squares it will be divisible by 4. *This is evident from Lemma* 1.

Lemma 4. If a Number consisting of two odd Squares be divided by 4 it will leave a Remainder of 2, by the 2d *Lemma.*

Lemma 5. If a Number consisting of an even and an odd Square be divided by 4 it will leave a Remainder of 1. *This Lemma is evident from the* 1st *and* 2d.

Con-

Consectary.

Hence it follows that any Number which divided by 4 leaves a Remainder of 3; as 3, 7, 11, 15, 19, &c. cannot be composed of two Integer Squares.

Scholium.

Moreover, I say that any such Number cannot be compos'd of two square Fractions. For, supposing it can, let such Fractions be equal to $\frac{yy}{aa}$ & $\frac{xx}{ee}$; their Sum will be $\frac{yyee+xxaa}{aaee}$ ($=4m+3$, by Hypothesis). 'Tis manifest the Denominator $aaee$ is a square, which must be odd or even.

Case 1. Suppose it odd and $=4nn+4n+1$; therefore $\frac{yyee+xxaa}{4nn+4n+1}=4m+3$, and $yyee+xxaa=16mnn+16mn+4m+12nn+12n+3$.

Now, since the latter Part of this Equation being divided by 4 leaves a Remainder of 3, it cannot be compos'd of two Integer Squares (by the foregoing Consect.) But the first Part of the Equation is, according to the Supposition, composed of two Integer Squares; consequently the Supposition, in this 1st *Case*, is absurd.

Case 2. Again let the Denominator $aaee$ be suppos'd even; then the Numerator $yyee+xxaa$ being a Multiple of $aaee$ (for the Fraction is equal to an Integer, *viz.* $=4m+3$) the Terms of the Numerator will be each of them an even Square too, and consequently (by *Lemma* 1) divisible by 4 (for, if one of the Terms was odd, and the other even, or if both of them were odd, their Sum would not be divisible by 4, by *Lemma*'s 5th and 4th, nor consequently by the Denominator or even Square $=aaee$); Let the Numerator and Denominator be therefore divided by 4; and, if the Denominator of the Fraction thence had be even, divide the Numerator and Denominator of this last Fraction by 4; and so proceeding 'till you have a Fraction whose Denominator is odd, this Fraction will be equal to the Former, and its Numerator will likewise consist of two Integer Squares, and its Denominator of one Inte-

Integer odd Square. Now this Fraction being $= 4m + 3$; if each Part of this Equation be multiplied by the Denominator, you'll have an Equation whose first Part will consist of two Integer Squares, and the second Part will not consist of two Integer Squares, by the 1st Case, which, amounting to a Contradiction, proves the Absurdity of the Supposition in this 2d Case.

By the foregoing *Consect.* any Number which being divided by 4, leaves a Remainder of 3, cannot be divided into two Integer Squares.

And, by the first and second Cases of this *Schol.* it cannot be divided into two square Fractions.

Consequently it cannot be divided into any two Rational Squares. *w. w. D.*

Question 1.

To divide a given Number which is compos'd of one or two known Squares, suppose $dd =$ the greater, and $bb =$ the lesser, into two other Squares.

☞ *Here Note that, if the given Number be* $=$ dd, *viz. a square Number, then* $bb = 0$.

Solution.

1. Take two * unequal Numbers $s =$ the greater, and $r =$ the lesser, with this * Caution, *viz.* that s be not in Proportion to r as $d + b$ to $d - b$; and for the Side of the first Square sought put $ra + b$

2. And for the Side of the second Square sought put $sa \backsim d$

3. Then the first Square sought is $= rraa + 2rba + bb$.

4. And the second Square sought is $= ssaa - 2sda + dd$.

5. Consepuently the Sum of those Squares is $rraa + ssaa + 2rba - 2sda + bb + dd$.

6. Which Sum must be equal to $bb + dd$. Hence the following Equation *viz*: $ss + rr : \times aa + : 2rb - 2sd : \times a + dd + bb = dd + bb$

7. Which Equation, after due Reduction, gives $a = \frac{2sd - 2rb}{ss + rr}$

* *For if* s *be* = r, *or if* s .. r :: d + b .. d − b, *then the squares that wou'd be found by the Canons wou'd be the same with the given ones.*

8. There-

8. Therefore, from the 7th and 1st Steps, the Side of the first Square sought is now known and found

$$=\frac{2rsd+ssb-rrb}{ss+rr}.$$

9. And, from the 7th and 2d Steps, the Side of the second Square sought is likewise known and found

$$=\frac{:ssd-rrd:\backsim 2rsb}{ss+rr}.$$

But if $b=0$, then the 8th and 9th Steps will become respectively,

10. $\frac{2rsd}{ss+rr}=$ Side of the first Square sought; and

11. $\frac{ssd-rrd}{ss+rr}=$ Side of the second Square sought.

From the 10th and 11th Steps ariseth the following Canon for dividing a given square Number into two other square Numbers.

Canon.

Take any two unequal Numbers, multiply severally the double of the Product of their Multiplication, and the Difference of their Squares by the Side of the given Square; then divide those Products severally by the Sum of the Squares of the two Numbers first taken, and the Quotients shall be the Sides of the two Squares sought.

And the 8th and 9th foregoing Steps give this Canon for dividing a given Number compos'd of two known Squares into two other Squares.

Canon.

Take any two unequal Numbers, with this Caution, that the greater may not have the same Proportion to the lesser, as the Sum of the Sides of the two Squares given hath to the Difference of the same Sides. Multiply the double Product of the Multiplication of those two Numbers first taken by each of the said two given Sides, and reserve the Products: Multiply also the Difference of the Squares of the said two Numbers first taken by each of the said two Sides given, and reserve these Products. Then add the

greater of the two first reserv'd Products to the lesser of the two latter, and reserve the Sum for a Dividend. Take also the Difference between the lesser of the two first Products and the greater of the two latter for a second Dividend: Lastly divide severally the said Dividends by the Sum of the Squares of the two Numbers first taken: So shall the Quotients be the Sides of the two Squares sought.

Quest. 2. To find two square Numbers whose Difference shall be equal to a given Number, suppose $= bc$.

Solution.

1. Let some Number whose Square is less than the given Difference be represented by b.
2. For the Side of the lesser Square sought put a.
3. For the Side of the greater Square sought put $a+b$
4. Then the lesser Square is $= aa$.
5. And the greater Square is $= aa + 2ab + bb$.
6. And the Difference of those Squares is $2ab + bb$.
7. But the said Difference must be equal to the given Difference; therefore $2ab + bb = bc$.
8. Which, after due Reduction, makes known the Value of the Side of the lesser Square; *viz.* $a = \frac{1}{2}c - \frac{1}{2}b$.
9. And from the 8th and 3d Steps the Value of the Side of the greater Square is also discover'd; *viz.* $a + b = \frac{1}{2}c + \frac{1}{2}b$.

Canon.

Take two such unequal Numbers that the Product of their Multiplication may be equal to the given Difference; then half the Sum and half the Difference of those two Numbers shall be the Sides of the two Squares sought.

Quest. 3. To find two such square Numbers that, if to the Product of their Multiplication a given Number $= d$ be added, the Sum may be a Square.

Solution.

1. For one of the Squares sought take any known square Number, which may be represented by bb.
2. And for the other Square sought put aa.
3. Then the Product of their Multiplication is $bbaa$.
4. To which Product the given Number d being added the Sum is $bbaa + d$.

5. Which

5. Which Sum must be equal to a Square, the Side whereof may be suppos'd to be ba — any known Number greater than $\sqrt{d}$; suppose $ba - c$; then the Square of $ba - c$, that is $bbaa - 2bca + cc$ being equated to $bbaa + d$, this Equation ariseth, *viz.* $bbaa + d = bbaa - 2bca + cc$.

6. Whence, after due Reduction, $a = \frac{cc - d}{2bc}$;

From the Premises ariseth this following

Canon.

For one of the squares sought take any square Number; then from any square Number substract the given Number; and divide the Remainder by the double of the Product made by the Multiplication of the Sides of those two Squares: So the Quotient shall be the Side of the other Square sought.

Quest. 4. To find two Numbers in a given Ratio, suppose the greater to the lesser as b to c; and that either of them added to the Square of the other Number may make a Square.

1. Suppose the greater Number sought $= 4ae$.
2. And the lesser of them $= a - e$.

Then 'tis manifest that the greater Number added to the Square of the lesser (*viz.* $aa - 2ae + ee + 4ae = aa + 2ae + ee$) is $= \square$ (whose Root is $a + e$) whereby one Part of the *Question* is satisfied.

3. Now (*per Question*) $b \,..\, c :: 4ae \,..\, a - e$.
4. Therefore $ba - be = 4cae$.
5. By Transposition and Division $e = \frac{ba}{b + 4ca}$
6. From 1st and 5th Steps the greater Number sought is $= \frac{4baa}{b + 4ca}$
7. From the 2d and 5th Steps the lesser sought Number is $= \frac{4caa}{b + 4ca}$.

But, besides the Limitations between the 2d and 4th Steps, the *Question* likewise requires that the lesser Number added to the Square of the greater should be a Square; *viz:*

 8. That

8. That $\frac{16bba^4}{bb+8bca+16ccaa}+\frac{4caa}{b+4ca}=$ $\frac{16bba^4+16c^2a^3+4bcaa}{bb+8bca+16ccaa}$ ſhould be $=\square$; in order to which

9. Suppoſe the Side of this Square $=\frac{4ba^2 \backsim da}{b+4ca}$ (d being taken $=$ any Number $\sqsubset 2\sqrt{bc}$.)

10. From 8th and 9th Steps $16b^2a^4+16c^2a^3+4bca^2$ $=16b^2a^4-8bda^3+ddaa$.

11. The Equation in the 10th Step, being reduc'd, gives a $=\frac{dd-4bc}{8bd+16cc}$.

12. a being thus found, two Numbers ſuch as are required will be given by the 6th and 7th Steps.

Example.

Let b be $=3$, and $c=2$: And ſuppoſe $d=10$; then $a\left(=\frac{dd-4bc}{8bd+16cc}\right)=\frac{100-24}{240+64}=\frac{1}{4}$; wherefore the greater Number ſought $\left(=\frac{4baa}{b+4ca}\right)=\frac{3}{4}\div:3+\frac{8}{4}:=\frac{3}{20}$.

And the leſſer ſought Number $\left(=\frac{4caa}{b+4ca}\right)=\frac{1}{2}\div 5=$ $\frac{1}{10}=\frac{2}{20}$.

Proof.

$3 \cdot\cdot 2 :: \frac{3}{20} \cdot\cdot \frac{2}{20}$.

Alſo $\frac{4}{400}+\frac{3}{20}=\frac{64}{400}$ is a Square (whoſe Root is $\frac{8}{20}$).
And $\frac{9}{400}+\frac{2}{20}=\frac{49}{400}$ is a Square (whoſe Root is $\frac{7}{20}$).

PART

PART. XVIII.

Of the Alternations and Combinations of Quantities.

Chap. I. Of the Alternations of QUANTITIES.

Definition.

ALternation is a Word used by Mathematicians for the different Changes, or Alterations of order in any Number of Things propos'd taken one by one, two by two, or three by three, &c.

Lemma.

The Number of Alternations of m Things $a^p\ b^q\ c^r$, &c. taken n by n, is equal to the Number of Alternations of the $m-1$ Things $a^{p-1}\ b^q\ c^r$, &c.

$+$ the Number of Alternations of the $m-1$ Things $a^p\ b^{q-1}\ c^r$, &c.

$+$ the Number of Alternations of the $m-1$ Things $a^p\ b^q\ c^{r-1}$, &c.

$+$ &c.

Taken $:n-1:$ by $:n-1:$

Demonstration.

It is evident that by placing the Thing a in any determin'd Place, as suppose in the first Place, of every Alternation which can be made of the $m-1$ Things $a^{p-1}\ b^q\ c^r$, &c. take $n-1$ by $n-1$; that each Alternation, by such Position of a produc'd, will consist of n Things; and that all these Alternations, in Number equal to the Number of Alternations of the $m-1$ Things $a^{p-1}\ b^q\ c^r$, &c. taken $n-1$ by $n-1$, are all the Alternations that can be made of the

m Things

m Things $a^p\ b^q\ c^r$, &c. taken n by n, which will have any a in the first Place of each Alternation of them.

For the same Reasons, the Number of Alternations of the m Things $a^p\ b^q\ c^r$, &c. taken n by n, which will have b in the first Place of each of them is equal to the Number of Alternations of the $m-1$ Things $a^p\ b^{q-1}\ c^r$, &c. taken $n-1$ by $n-1$.

Also the Number of Alternations of the m Things $a^p\ b^q\ c^r$, &c. taken n by n which will have c in the first Place of each of them is equal to the Number of Alternations of the $m-1$ Things $a^p\ b^q\ c^{r-1}$, &c. taken $n-1$ by $n-1$.

&c.

Wherefore the Number of Alternations of the m Things $a^p\ b^q\ c^r$, &c. taken n by n, is equal to the Number of Alternations of the $m-1$ Things $a^{p-1}\ b^q\ c^r$, &c.

$+$ the Number of Alternations of the $m-1$ Things $a^p\ b^{q-1}\ c^r$, &c.

$+$ the Number of Alternations of the $m-1$ Things $a^p\ b^q\ c^{r-1}$, &c.

$+$ &c.

Taken $n-1$ by $n-1$. *Q. E. D.*

Scholium.

In order to find the Number of Alternations of m Things $a^p\ b^q\ c^r$, &c. taken one by one, two by two, or three by three, &c. by the help of our *Lemma*: Let the Number of the Indices in the said m Things, which are each not less than 1, 2, 3, 4, &c. be suppos'd equal to A, B, C, D, &c. respectively: Then the Number of the Indices in the said m Things which are each equal to 1, 2, 3, &c. is equal to $A-B$, $B-C$, $C-D$, &c. respectively. Then

1. The Number of Alternations of m Things $a^p\ b^q\ c^r$, &c. taken one by one is manifestly $=A$.

2. The Number of Alternatiens of m Things $a^p\ b^q\ c^r$, &c. taken two by two, is by our *Lemma*, $=:A-B:\times$ Number of Alternations of $m-1$ Things, wherein $A-1$ is equal to the Number of all the Indices

$+B\times$ Number of Alternations of $m-1$ Things wherein A is equal to the Number of all the Indices.

Taken one by one

$=:A-B:\times:A-1:+B\times A$, by Parag. 1, $=A\times:A-1:+B$.

3. The

3. The Number of Alternations of m Things $a^p b^q c^r$, &c. taken there by three, is, by our *Lemma*, $= : A - B : \times$ Number of Alternations of $m-1$ Things, wherein $A-1$, and B are equal to the Number of the Indices which are each not leſs than 1 and 2 reſpectively

$+ : B - C : \times$ Number of Alternations of $m-1$ Things wherein A and $B-1$ are equal to the Number of the Indices which are each not leſs than 1 and 2 reſpectively

$+ C \times$ Number of Alternations of $m-1$ Things, wherein A and B are equal to the Number of the Indices which are each not leſs than 1 and 2 reſpectively.

Taken two by two.

$$= \overline{A-B} \times : \overline{A-1} \times \overline{A-2} + B : + \overline{B-C} \times : A \times \overline{A-1} + \overline{B-1} : + C \times : A \times \overline{A-1} + B :, \text{ by Parag. 2}, = A \times \overline{A-1} \times \overline{A-2} + 3AB - 3B + C.$$

4. The Number of Alternations of m Things $a^p b^q c^r$, &c. taken four by four is by our *Lemma* $=$

$: A - B : \times$ Number of Alternations of $m-1$ Things, wherein $A-1$, B and C are equal to the Number of the Indices which are each not leſs than 1, 2 and 3 reſpectively

$+ : B - C : \times$ Number of Alternations of $m-1$ Things, wherein A, $B-1$, and C are equal to the Number of the Indices which are each not leſs than 1, 2 and 3 reſpectively

$+ : C - D : \times$ Number of Alternations of $m-1$ Things wherein A, B and $C-1$ are equal to the Number of the Indices which are each not leſs than 1, 2 and 3 reſpectively

$+ D \times$ Number of Alternations of $m-1$ Things wherein A, B and C are equal to the Number of the Indices which are each not leſs than 1, 2 and 3 reſpectively

Taken three by three

$$= \overline{A-B} \times : \overline{A-1} \times \overline{A-2} \times \overline{A-3} + 3B \times \overline{A-1} - 3B + C :$$

$$+ \overline{B-C} \times : A \times \overline{A-1} \times \overline{A-2} + 3A \times \overline{B-1} - 3 \times \overline{B-1} + C :$$

$$+ \overline{C-D} \times : A \times \overline{A-1} \times \overline{A-2} + 3AB - 3B + \overline{C-1} :$$

$$+ D \times : A \times \overline{A-1} \times \overline{A-2} + 3AB - 3B + C :, \text{ By Parag. 3.}$$

$$= A \times \overline{A-1} \times \overline{A-2} \times \overline{A-3} + 6A^2B - 18AB + 9B + 3B^2 + 4CA - 4C + D.$$

&c.

Example.

Example.

Let it be required to find the Number of Alternations of $a^3\,b^3\,c^2$ (that is of the 8 Things *aaabbbcc*) taken 4 by 4.

Here A is $=3$, $B=3$, $C=2$, and $D=0$; wherefore, by the 4^{th} Parag. of our Scholium, $3\times2\times1\times0\ (0)+6\times3^2\times3\ (162)-18\times3\times3\ (-162)+9\times3\ (27)+3\times3^2\ (27)+4\times2\times3\ (24)-4\times2\ (-8)+0=70$ is the Number of Alternations of $a^3\,b^3\,c^2$ taken four by four.

Corollary.

From what hath been said in this *Scholium*, it is plain that the Number of Alternations of *m* Things different from each other, as *a b c d*, &c. taken *n* by *n* is (because A, in this Case, is = to *m*, and B, C, D, &c. are each equal to 0) $=A\times\overline{A-1}\times\overline{A-2}\times\overline{A-3}\times\overline{A-4}\times$ &c. continued to *n* places $=m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times$ &c. continued to *n* places.

Examples.

1. Let it be required to find the Number of Alternations of four Things different from each other, as *a b c d*, taken four by four.

Here $m=A$ is $=4$, and $n=m=4$; wherefore, by our *Corollary*, $4\times:4-1:\times:4-2:\times:4-3:=24$ is the Number of Alternations of the four Things *a b c d*, different from each other, taken four by four.

abcd, *bacd*, *cabd*, *dabc*,
abdc, *badc*, *cadb*, *dacb*,
acbd, *bcad*, *cbad*, *dbac*,
acdb, *bcda*, *cbda*, *dbca*,
adbc, *bdac*, *cdab*, *dcab*,
adcb, *bdca*, *cdba*, *dcba*.

2. Let it be required to find the Number of Alternations of the five Things *a b c d e* different from each other taken two by two.

Here $m=A$ is $=5$, and $n=2$; therefore, by our *Corollary*, $5\times4=20$ is the Number of Alternations, of five Things different from each other taken two by two.

ab, *ba*,

ab, ba, ca, da, ea,
ac, bc, cb, db, eb,
ad, bd, cd; dc, ec,
ae, be, ce, de, ed.

☞ Note, *When* n *is* $=$ m, *or* $=$ m $-$ 1 *the Number of Alternations of* m *Things* $a^p b^q c^r$, *&c. taken* n *by* n, *that is* m *by* m, *or* m $-$ 1 *by* m $-$ 1 *is* $=$

$$\frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times\overline{m-5}\times\overline{m-6}\times\overline{m-7}\times\overline{m-8}\times\overline{m-9}\times \text{\&c. continued to } m \text{ or } m-1 \text{ Places}}{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\text{\&c.}\times q\times\overline{q-1}\times\overline{q-2}\times\overline{q-3}\times\text{\&c.}\times r\times\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times\text{\&c.}\times\text{\&c. continued to } p,\ q,\ r,\ \text{\&c. Places respectively}}$$

, *which is a more regular and simple Series than those that are, or may be, exhibited by the former Method.*

CHAP. II.

Of the Combinations *of* Quantities.

Definition.

THE several Ways or different Cases of taking any required Number of Things propos'd, without regarding their Order or Places, are called the Combinations of those Things.

Scholium 1.

From the Nature of Alternations and Combinations rightly consider'd and compar'd, it will appear that the Number of Alternations of *m* Things *a b c d e*, &c. different from each other, taken *n* by *n*, is equal to the Number of Combinations of the said *m* Things taken *n* by *n* multiplied by the Number of Alternations of *n* Things different from each other, taken *n* by *n*: But the Number of Alternations of the said *m* Things taken *n* by *n* is, by the *Corollary* in the last Chap. $= m\times:m-1:\times:m-2:\times:m-3:\times$ &c. continued to *n* Places: And the Number of Alternations of *n* Things different from each other taken *n* by *n* is, by the said *Corollary*, $= n\times:n-1:\times:n-2:\times:n-3:\times$ &c. conti-

 nued

nued to n Places; therefore the Number of Combinations of the said m Things taken n by n is $=$

$$\frac{m \times : m-1 : \times . m-2 : \times : m-3 : \times : m-4 :}{n \times : n-1 : \times : n-2 : \times : n-3 : \times : n-4 :} \times, \&c.$$ each Series continued to n Places or Terms.

Example.

Let it be required to find the Number of Combinations of the five Things $a\,b\,c\,d\,e$ different from each other taken three by three.

Here m is $= 5$, and $n = 3$; therefore $\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$ is the Number of Combinations required.

abc, acd, ade, bcd, bde, cde.
abd, ace, bce,
abe,

Scholium 2.

It is easier to find the Number of Alternations of m Things $a^p\,b^q\,c^r$, &c. taken n by n by the first Method in the last Chap. than to find the Number of their Combinations by the like Method: And, when n is $=$ any great Number, it would require an impracticable Calculation to find the Number of their Combinations or Alternations by that Method: Wherefore,

In order to find the Number of Combinations and of Alternations of m Things $a^p\,b^q\,c^r$, &c. taken n by n, if n be $= m$ or $= m-1$, then the Number of Alternations required may be easily had by the Remark in the latter Part of the last Chap. But if n be less than $m-1$, and that the Number of Combinations of such Things so taken is not many; then write down all these Combinations distinctly, and then the Number of Alternations of each of those Combinations taken n by n will be found by the said Remark.

In order to which it will be convenient, in some Cases, to observe that the Number of Combinations of m Things taken n by n (n being less than m) is $=$ Number of Combinations of the said m Things taken $m-n$ by $m-n$; for the Things composing each Combination of these taken from the said m Things leave the Things composing each Combination of the former. Thus

Suppose

Suppose it was required to find the Number of Combinations and of Alternations of the eight Things $a^3\ b^3\ c^2$ taken six by six.

By the above Observation the Number of Combinations required is = that of the Combinations of the 8 Things $a^3\ b^3\ c^2$ taken 8 — 6 by 8 — 6, *viz.* 2 by 2: And the Combinations themselves of the said 8 Things taken 2 by 2, are a^2, ab, ac, b^2, bc, and c^2; consequently

$$\text{From } a^3\ b^3\ c^2 \text{ take } \left\{\begin{matrix} c^2 \\ bc \\ b^2 \\ ac \\ ab \\ a^2 \end{matrix}\right\} \text{Remains } \left\{\begin{matrix} a^3\ b^3 \\ a^3\ b^2\ c \\ a^3\ b\ c^2 \\ a^2\ b^3\ c \\ a^2\ b^2\ c^2 \\ a\ b^3\ c^2 \end{matrix}\right\}$$

which Combinations, being in Number six, are all the Combinations which the eight Things $a^3\ b^3\ c^2$ taken six by six are capable of.

Next it will be proper to take Notice that four, and no more, of those Combinations, *viz.* the 2d, 3d, 4th, and 6th have the same Indices; and therefore the Number of Alternations of one of them is equal to the Number of Alternations of any other of them.

Now, (by the Remark in the latter Part of the last Chap.) the Number of Alternations of the first Combination, *viz.* $a^3\ b^3$ taken 6 by 6 is $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1} = 20.$

Also, the Number of Alternations of the second Combination $a^3\ b^2\ c$ taken 6 by 6 is $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1 \times 1} = 60$ = Number of Alternations of $a^3\ b\ c^2$, as also of $a^2\ b^3\ c$, and of $a\ b^3\ c^2$, severally, taken 6 by 6.

And the Number of Alternations of the six Things in the fifth Combination, *viz.* of $a^2\ b^2\ c^2$ taken 6 by 6 is $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1} = 90.$

Consequently the Number of Alternations of the eight Things $aaa\ bbb\ cc$ taken six by six is $= 20 + 60 \times 4 + 90 = 350.$

I believe it was from a due Consideration of the like Examples with this, that Col. Thornycroft *deduc'd his Method or Theorem for finding the Numbers of Combinations and Alternations of any Quantities expos'd taken* n *by* n*; to which Method, it being inserted in* Phil. Transf. N° 299. *I refer such Mathematicians as have a Mind to know more of this Doctrine.*

PART XIX.

Of *Magick* Squares.

QUEST. 1. The Numbers, 1, 2, 3, 4, 5, 6, 7, 8, and 9 being given; tis required to place them in a Magick Order; *viz.* in a square Form, so as counting each Rank up and down, as also from one Hand to the other, and Diagonal-wise; that those Ranks may be equal to each other.

Suppose it done and represented in its proper Form by the following Symbols thus plac'd; *viz.*

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

The Sum of the propos'd Figures is 45; and $\frac{45}{3}=15$ is $=$ each Side or Rank $=$ (suppose) s.

Then	1	$a+e+i=s$	By the Nature of the *Quest.*
	2	$b+e+h=s$	
	3	$c+e+g=s$	
$1+2+3$	4	$a+b+c+3e+i+h+g=3s.$	
	5	$a+b+c+i+h+g=2s$, *per Quest.*	
$4-5$	6	$3e=3s-2s=s.$	
$6\div\overline{3}$	7	$e=\frac{s}{3}=\frac{15}{3}=5.$	

The Value of e being thus found to be 5, there remain eight Figures more, *viz.* 1, 2, 3, 4, 6, 7, 8 and 9; But which of those is equal to any corner Letter, as suppose a, is to be further sought.

Beginning therefore with the least Number 1; I say the corner Letter a, and consequently any corner Letter as c, i, g, cannot be equal to it: For, if a was $=1$, then i should be $=9$; and $b+c=15-1=14$ as also $=d+g$; But there remain no two Numbers (after 5, 1 and 9) whose Sum is

is 14 but 6 and 8; therefore, if either of these Figures were $=b$, the other would be $=c$; and, then no Figure would remain for the Value of either d or g: Wherefore a is not $=1$; neither is i, nor, consequently, any corner Letter equal to 1 or 9.

2 May be $=a$, as will appear farther on.

3 Cannot be equal to a; for, if it were, then i should be $=7$; and $b+c=15-3=12$, as also $=d+g$: But there remain no two Numbers (after 5, 3 and 7) whose Sum is 12 but 8 and 4, which cannot answer to b and c, and d and g; wherefore a, or any other corner Letter is not $=3$; neither is i, nor consequently any other corner Letter $=7$.

From what hath been said 'tis plain that (if the *Question* propos'd is capable of being solv'd) the corner Letters are all equal to even Numbers; wherefore, placing one of them, as 2 for a, i will be $=8$, and c must be either equal to 4 or 6; let it (*viz.* c) be $=4$; then $g=6$; $b=9$; $d=7$; $f=3$; and $h=1$: and so the Square is compleated as required.

2	9	4
7	5	3
6	1	8

But if c were $=6$ (a being $=2$); then $g=4$; $b=7$; $d=9$; $f=1$; and $h=3$: And then the Square will stand, thus

2	7	6
9	5	1
4	3	8

&c.

Quest. 2. The Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16 being given; 'tis required to place them in a Magick Order, *viz.* in a square Form so as counting each Rank from one Hand to the other, as also up and down, and Diagonal-wise, that those Ranks may be equal to each other.

Suppose it done and represented in its proper Form by the following Symbols thus plac'd; *viz.*

a b c d

a	b	c	d
e	f	g	h
i	k	l	m
n	o	p	q

The Sum of the propos'd Figures is 136; and $\frac{136}{4}=34$ is $=$ each Rank $=$ (suppose) s: Then, by the Nature of the *Question*,

1 $a+b+c+d+e+f+g+h=a+e+i+n+b+f+k+o$
2 $\therefore c+d+g+h=i+n+k+o$.
3 In like Manner $a+b+e+f=l+p+m+q$
4 $a+b+c+d+n+o+p+q=d+g+k+n+a+f+l+q$
5 $\therefore b+c+o+p=g+k+f+l$.
6 But the Sum of the two Parts of the 5th Equation is $=2s$; consequently either Part of it is $=s=34$
7 $a+b+c+d+n+o+p+q=a+e+i+n+d+h+m+q$
8 $\therefore b+c+o+p=e+i+h+m=s$, *per* 6°.
9 $a+b+c+d+n+o+p+q=b+f+k+o+c+g+l+p$
10 $\therefore a+d+n+q=f+k+g+l=s$, *per* 6°.
11 $\therefore a+d+n+q=d+h+m+q$
12 $\therefore a+n=h+m$
13 In like Manner $d+q=e+i$
14 Also $n+q=b+c$
15 And $a+d=o+p$
16 $e+i+h+m=i+k+l+m$, *per* 8°.
17 $\therefore e+h=k+l$
18 Also $c+p=f+k$
19 Also $b+o=g+l$
20 And $f+g=i+m$
21 $f+k+g+l=a+f+l+q$ *per* 6°.
22 $\therefore k+g=a+q$
23 Likewise $f+l=d+n$.

Having thus far proceeded, and the Question propos'd being (probably) capable of a great many different Solutions; 'tis to be presumed that a may be equal to any of the given Numbers: Beginning therefore with the least of them, *viz.* 1, and putting a equal to it *viz.* $=1$; the next Thing to be done is to find the Value of another corner Letter as n.

n cannot be equal to 2; for, if it was; then $h+m$ being (*per*

(*per* 12°.) $= a + n$ $(= 1 + 2)$ should be $= 3$; but there are no remaining two Numbers of the given ones whose Sum is 3; consequently n cannot be $= 2$ (a being $= 1$.)

Neither can n be equal to 3: For supposing $n = 3$, then $b + m$ being (*per* 12°.) $= a + n$ $(= 1 + 3)$ shou'd be $= 4$: But there are no remaining two Numbers of the given ones whose Sum is 4; therefore, &c.

But (for ought can be seen yet) n may be equal to 4; putting therefore $n = 4$; then $b + m$ $(= a + n$, *per* 12°. $= 1 + 4) = 5$; that is b, m are equal to 2, 3 which are the only two Numbers remaining whose Sum is 5; and therefore $d + q = 34 - 5 = 29 = e + i$ (*per* 13); that is d, q are equal to 13, 16 or 14, 15; and accordingly e, i are equal to 14, 15 or 13, 16 only; for no other couple amounts to 29.

Let us now see what the Consequence is of putting $q = 13$; then d is $= 16$: And then the Square may be designed, in part, thus

1	*b*	*c*	16
	f		
14, 15			2, 3
	k	*l*	
4	*o*	*p*	13

The four corner Letters being thus design'd; and consequently e, i equal to 15, 14, as also b, m being equal to 2, 3, it is manifest f cannot be equal to 5, 6 or 7; for, if it was, l should be equal to 15, 14 or 13 which are Numbers already dispos'd off: But (perhaps) f may be, and therefore suppose it, $= 8$, and then l will be $= 12$.

Again $g + k = 14$; and there remain no two Numbers whose Sum is 14, only 5 and 9: But $k + l$ or $k + 12$ is likewise (*per* 17°) $= e + b$, *viz.* equal to 16 $(2 + 14)$, 17 $(2 + 15$ or $3 + 14)$, or 18 $(3 + 15)$; consequently k is equal to 4, 5 or 6 (Not equal to 9)); and therefore it (*viz.* k) is $= 5$.

$k + l$ $(= 5 + 12)$ being thus found $= 17$ must likewise be $= e + b$ (*per* 17°.), which may be effected two Ways; *viz.* by putting $e = 15$, and then b will be $= 2$; or, putting $e = 14$, b will be $= 3$: Let us chose the former; and then i will be $= 14$, and m will be $= 3$: And then the Square will be farther designable, thus

1 *b*

1	*b*	*c*	16
15	8	9	2
14	5	12	3
4	*o*	*p*	13

It remains to dispose of the four Numbers 6, 7, 10 and 11 instead of b, c, o and p, so as $b+c$ may be $=17$, as also $o+p=17$, which may be done by coupling 6, 11 as also 7, 10: But $c+p$ must be (by 18°) $=k+f=13$, which will be effected by $6+7$: From whence p being $=6$, c will be $=7$; and then $o=11$, and consequently $b=10$: And then the Square will be fully compleated, thus

1	10	7	16
15	8	9	2
14	5	12	3
4	11	6	13

Or putting $p=7$; then $c=6$, then $o=10$, and $b=11$: And then the Square will stand, thus

1	11	6	16
15	8	9	2
14	5	12	3
4	10	7	13

&c.

Note, Book II. begins Page 321. Signature * Y

A TREATISE

A

TREATISE

OF

ALGEBRA.

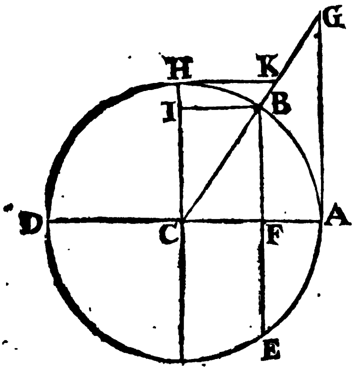
BOOK II.

Note, *Besides the Characters explain'd in* Book I. *Pages* 2, 3, *and* 4. *these following are added*; viz.

r or R	Signifies	Radius.
S		Sine.
Σ		Co-sine.
$\int$		Secant.
[illegible]		Co-secant.
T		Tangent.
τ		Co-tangent.
v		Versed Sine.
$\angle$		Angle.
$\llcorner$		Right-angle.
$\perp$		Perpendicular.
$\parallel$		Parallel.
—		Straight Line.
$\triangle$		Triangle.
$\square$		Square.
$\sqsubset\!\sqsupset$		Oblong, or Parallelogram.

PART I.
Of Plane Trigonometry.

DEFINITIONS.



1. A Circle is suppos'd to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds; &c. Any Portion of whose Circumference is called an Arch, and is measured by the Number of Degrees it contains.

2. A Chord or Subtense is a straight Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

3. A Sine (or Right-sine) is a straight Line drawn from one End of an Arch perpendicular to that Diameter passing thro' the other End; or it is half the Chord of twice the Arch; so BF is the Sine of the Arcs BA, BD. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greatest of all Sines, the Sine of an Arc greater than a Quadrant being less than the Radius.

4. The Difference of an Arc from a Quadrant, whether it be greater or less, is called its Complement; so HB is the Complement of the Arcs BA, BD; BI is the Sine of that Complement, aud therefore it is called the Co-sine, or Sine-Complement af the Arcs BA, BD.

5. The Secant of an Arc is a straight Line drawn from the Center thro' one End of the Arc till it meet with the Tangent, which is a straight Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arc; so CG is the Secant, and AG the Tangent of the Arcs BA, BD: And CK is the Co-secant, and HK the Co-tangent of the said Arcs.

6. A

6. A Versed Sine is the Segment of the Diameter intercepted between the Arc and its Sine: Thus F A is the Versed Sine of the Arc B A, and F D of the Arc B D.

7. Whatever Number of Degrees an Arc wants of a Semi-Circle is called its Supplement.

8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-sine; thus C F is = I B.

9. If an Arc be greater or less than a Quadrant, the Sum or Difference of the Radius and Co-sine is equal to the Versed-sine.

In a Triangle are six Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a plane Triangle, the other three may be found either mechanically, by the help of a Scale of equal Parts and Line of Chords, or by an Arithmetick Calculation, if, supposing the Radius divided into any Number of equal Parts, we know how many of those equal Parts are in the Sine Tangent, or Secant of any Arc propos'd: The Art of inferring which is called *Trigonometry*, and is either Plane or Spherical.

A Method of computing the natural Sine, Tangent or Secant of any Arc immediately, from the Length of the Arc being given.

The Length of any Arc is readily obtain'd from the Ratio of the Diameter of the Circle to its Circumference, exhibited by *Van Ceulen*, since prolong'd and confirm'd by others, which is As 1 To 3 . 14159265358979323 8, &c. [*See the first* Schol. *to* Sol. 2. Prob. 3. Chap. 3. Part III.] This Number, the Radius being 1, is the just Length of the Semi-circle or Arc of 180°; whence any less Arc is easily got by Division. Thus the Number of Minutes in 180° is 10800; by which 3 . 14159 &c. being divided, gives . 000290888208665 72159 + for the Length of the Arc of 1 Minute, which being multiplied by the Number of Minutes contained in any other Arc, serves readily to give its Length. Hence, by Sir *Is. Newton*'s Series, publish'd by Dr. *Halley*, in *Phil. Trans.* N° 219, the Sine, Co-sine, Tangent, &c. of any Arc are had.

Thus if the Length of any Arc be put $= a$, and Radius $= 1$, then is the Natural

Sine $= a - 6)aaa - 20)aaB - 42)aaC - 72)aaD - 110)aaE$, &c. putting B, C, D, E, &c. for the second, third, fourth, fifth, &c. Terms. [*See* Schol. 1. *to* Sol. 1. Prob. 3. Chap. 3. Part III.]

Co-sine $= 1 - 2)aa - 12)aa\mathfrak{B} - 30)aa\mathfrak{C} - 56)aa\mathfrak{D} - 90)aa\mathfrak{E}$, &c. putting $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. for the second, third, fourth, fifth, &c. Terms. [*See* Schol. 2. *to the last mention'd* Prob.]

Tangent $= a + 3) a^3 + 15) 2a^5 + 315) 17 a^7 + 2835) 62 a^9 + 155925) 1382 a^{11} +$ &c.

Co-tangent $= a) 1 - 3) a - 45) a^3 - 945) 2 a^5 - 4725) a^7 - 93555) 2 a^9 - 638512875) 1382 a^{11} -$ &c.

Secant $= 1 + 2) a^2 + 24) 5 a^4 + 720) 61 a^6 +$ &c. &c.

Examples.

1. Let it be required to find the Sine and Co-fine of 5 Minutes.

$$
\begin{array}{r l}
.0002908882086657 2 = & \text{Arc of } 00^\circ.\ 01'. \\
5 & \\
\hline
.0014544410433286 0 = & \text{Arc of } 00^\circ.\ 05' = a.
\end{array}
$$

Now, *first*, for the Sine of 5 Minutes.

$$
\begin{array}{r r}
a = & .0014544410433 3 - \\
- 6) aaa = - & 51279 - \\
\hline
\text{The Sine of } 00^\circ.\ 05' = & .0014544405 3054
\end{array}
$$

2*dly*, For the Co-fine of 5 Minutes, or the Sine of 89°. 55′.

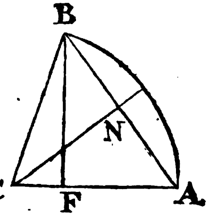
$$
\begin{array}{r r}
1 = 1 & \\
- 2) aa = - & .00000105769937 \\
- 12) aaB = + & 18 \\
\hline
\text{The Co-fine of 5 Min.} = & .99999894230081
\end{array}
$$

2. If the Sine of 29°. 55′ be required. The Number of Minutes contained therein is 1795, by which ·000290888 &c. being multiplied, make the Length of the Arc = ·522144334 55497 $= a$.

$$
\begin{array}{r r r}
a = & .52214433455497 & \\
- 6) aaa = - & & .02372577786621 \\
- 20) aaB = + & 32342352378 & \\
- 42) aaC = - & & 209943993 \\
- 72) aaD = & 794972 & \\
- 110) aaE = - & & 1970 \\
- 156) aaF = + & 3 & \\
\hline
& .52246776602850 - & .02372787732584 = \\
& .49873988870266 = & \text{the Sine of } 29^\circ.\ 55'.
\end{array}
$$

Since these Series converge the swiftest near the Beginning and End of the Quadrant; for raising a Table no more than the first or last 30 Degrees need be calculated. The rest are to be obtained from them by such Methods as shall be shewn farther on.

1. BF the Sine of an Arc being given; to find its Co-sine CF.



☞ N.B. *The Radius is always suppos'd to be given.*

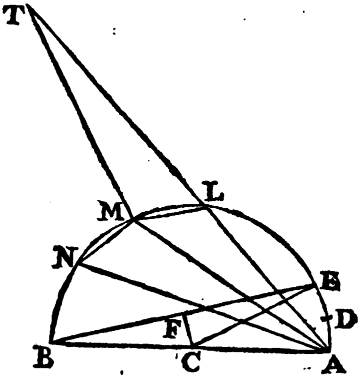
$CBq = BFq + CFq$ (by 47. 1 *Eucl. Elem.*)

Therefore $\surd : CBq - BFq : = CF$, by Transposition and Evolution; that is $\surd : Rq - Sq : = \Sigma$.

2. BF the Sine of an Arc being given; to find AN the Sine of half the Arc.

CF the Co-sine is known (by the first) and consequently FA: Then $\surd : BFq + FAq : = BA$ (by 47. 1. *Eucl. El.*) and $\frac{1}{2} BA = AN$ (by the third *Definition*); *i. e.* $\frac{1}{2} \surd : Sq + vq : = S \frac{1}{2}$ Arc.

3. To find the Sines of double, treble, quadruple, quintuple, &c. of any Arc, whose Sine is given successively.



If from A, the End of the Diameter AB of a Circle upon its Semi-Circumference AMB, you make the Arc AE equal to twice the given Arc AD, and AL, AM and AN equal to $n - 1$, n and $n + 1$ times the Arc AE (or $2n - 2$, $2n$ and $2n + 2$ times the Arc AD) respectively (n being = any Number not less than 1): I say, Radius is to double the Co-sine of the Arc AD, as the Sine of n times the Arc AD is to the Sum of the Sines of $n - 1$ times the Arc AD, and $n + 1$ times the Arc AD.

Draw the Chords AN, AM and AL, which AL produce to T, drawing MT = MA. Drawlikewise the Chords LM, MN, as also BE, and the Radius CE; and lastly CF ⊥ BE.

By 20. 3 *Eucl. El.* ∠ ECA = 2 ∠ EBA; therefore S ∠ EBA = S Arc DA; consequently the Co-sine of the Arc DA, or of ∠ EBA is BF, which is = FE; wherefore BE is = 2 Σ Arc DA. Again, the Angle LAM = MAN = EBA (by 27. 3 *Eucl. El.*) = CEB = LTM (by 5. 1. *Eucl. El.*) And the Sum of the Angles ANM and ALM is (by 22. 3 *Eucl. El.*) = 2 ∟ = ALM + TLM; wherefore the Angle ANM = TLM; also LM = NM: Consequently the Triangles ANM and TLM are similar and equal to one

a rother

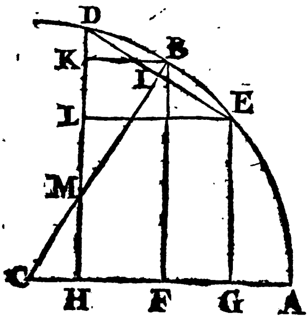
another: Whence $TL = AN$. The Triangles BCE and AMT are similar: Wherefore $CE \cdot\cdot BE :: AM \cdot\cdot AT :: \frac{1}{2} AM \cdot\cdot \frac{1}{2} AT$; that is, (by 3d and 4th preceding *Definitions*, and by what is above said) R (CE) $\cdot\cdot$ 2Σ Arc DA (BE) :: Sine of n times the Arc AD (or Sine of $\frac{1}{2}$ Arc AM, which is $\frac{1}{2} AM$) $\cdot\cdot$ Sine of $n-1$ times the Arc AD (or Sine of $\frac{1}{2}$ Arc AL, which is $\frac{1}{2} AL$) + Sine of $n+1$ times the Arc AD (or Sine of $\frac{1}{2}$ Arc AN, which is $\frac{1}{2} AN$) $= \frac{1}{2} AT$. *W.W.D.*

By this one Rule, after the first Sine, and consequently (by the first) double its Co-sine are obtain'd, the whole Work may (if need) be accomplish'd. Thus;

$R \cdot\cdot 2\Sigma DA ::$
$SDA \cdot\cdot S2DA$. *Note* DA, or Arc DA.
$S2DA \cdot\cdot SDA + S3DA$. Then $SDA + S3DA - SDA = S3DA$.
$S3DA \cdot\cdot S2DA + S4DA$. $S2DA + S4DA - S2DA = S4DA$,
&c.

And it has these Advantages, the two first Tems are invariable, the first being the Radius $= 1$, Division therefore is wholly excluded; the second being fix'd, a small Table of its Products, to 10, turns *Multiplication* to *Addition*.

4. Having given the Sines of all the Arcs, from the Beginning of a Quadrant, to any Part thereof distant from each other by an equal Interval; thence to find the Sines of all Arcs to the Double of that Part.



Let BF, the Sine of the Arc AB, be given; as also EG, the Sine of any Arc AE less than AB, and $EI = ID$, the Sine of the Arc BE (= Arc BD) the Difference of the Arcs AB and AE; and let DH, the Sine of the Arc AD, the Sum of the Arcs AB and BE, be required.

The Radius CB being drawn will intersect DH in M, and will be $\perp$ to, and bisect the straight Line DE in I. Draw IK and $EL \parallel$s the Radius AC.

Then the $\triangle$s DIK and DEL being similar, and $DE = 2DI$, DL must therefore be $= 2DK$. The Ld $\triangle$s CBF (CMH, DMI) and DIK are similar; wherefore $CB = R \cdot\cdot CF :: DI = EI \cdot\cdot DK$; therefore $R \cdot\cdot 2CF :: EI \cdot\cdot 2DK = DL = DH - EG$: Consequently $EG + \frac{EI \times 2CF}{R} = DH$.

Scholia.

Scholia.

1. If AB be 30 Degrees, then 2 CF ($= 2\,\Sigma\, 30^\circ = 2\, S\, 60^\circ$) will be $= R \times \sqrt{3}$, and $EG + EI \times \sqrt{3} = DH$: That is to say, the Sine of an Arc less than 30 Degrees, added to the Sine of its Defect $\times \sqrt{3}$, is = the Sine of an Arc so much exceeding 30 Degrees, as the other wanted of 30 Degrees.

Ex. gr. $S\, 19^\circ + S\, 11^\circ \times \sqrt{3} = S\, 41^\circ$.

2. If AB be 60 Degrees, then 2 CF will be = R (for $2\,\Sigma\, 60^\circ = 2\, S\, 30^\circ$ = Chord of 60° is = R, by 15. 4 *Eucl. El.*) and $EG + EI = DH$: That is to say, the Sine of an Arc less than 60°, added to the Sine of its Defect, gives the Sine of an Arc as much exceeding 60 Degrees.

Ex. gr. $S\, 41^\circ + S\, 19^\circ = S\, 79^\circ$; $\because S\, 79^\circ - S\, 41^\circ = S\, 19^\circ$;

Or $S\, 79^\circ - S\, 19^\circ = S\, 41^\circ$; that is, if from the Sine of an Arc exceeding 60 Degrees, the Sine of the Excess be subtracted, there will remain the Sine of an Arc wanting so much of 60 Degrees.

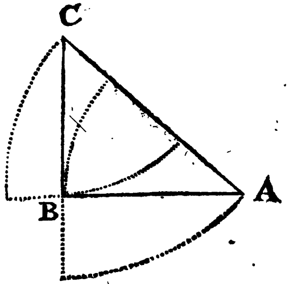
Having found the Sines and Co-sines, the Tangents, Secants, &c. may be found by the following Proportions.

The $\triangle$s CFB and CAG (*See* Fig. *in Page* 322) are similar; and the $\triangle$s CIB and CHK are also similar; therefore

$$\left.\begin{array}{l} CF \cdot\cdot CA :: FB \cdot\cdot AG \\ CF \cdot\cdot CA :: CB \cdot\cdot CG \\ CI(FB) \cdot\cdot IB(CF) :: CH \cdot\cdot HK \end{array}\right\}\text{that is}\left\{\begin{array}{l} \Sigma \cdot\cdot R :: S \cdot\cdot T. \\ \Sigma \cdot\cdot R :: R \cdot\cdot \text{ſ}. \\ S \cdot\cdot \Sigma :: R \cdot\cdot \tau. \end{array}\right.$$

Plane Trigonometry is solv'd by the Help of four fundamental Propositions, call'd *Axioms.*

Axiom 1.



In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then shall the other Leg CB or AB be the Tangent of the Angle opposite to it, and the Hypothenuse AC (or Side opposite to the Right-angle) its Secant (by *Definition* 5.)

But if the Hypothenuse AC be made the Radius of a Circle, then will

will the Legs (or Sides including the Right-angle) to wit CB and AB, be the Sines of the Angles opposite (by *Definition* 3.)

Upon this *Axiom* depends the Solution of the seven Cases of Right-angled Plane Triangles.

Note, That the three Angles of a Plane Triangle make two Right-Angles, or 180 Degrees, by 32. 1 *Eucl. El.*

For the more easy making the Proportions for the Solution of Right-angled Triangles, observe, that as different Sides are made Radius, so the other Sides acquire different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then say, As the Name of the Side given, Is to the Name of the Side required; So is the Side given, To the Side required.

But to find an Angle, one of the given Sides must be made Radius; then,

As the Side made Radius, Is to the other Side; So is the Name of the first Side (which is Radius) To the Name of the second Side; which fourth Proportional must be found among the Sines or Tangents, &c. to be determin'd by the Side made Radius, and against it is the Angle required.

The

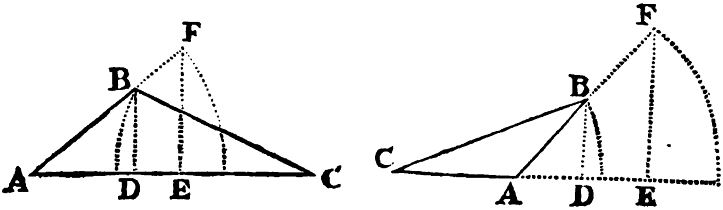
The Proportions for the Solution of the ſeven Caſes of Plane Right-angled Triangles. [*See the next foregoing* Fig.]

Given.	*Reqd.*	*Proportions.*	*Rad.*	*Caſe.*
AB A and C	BC	ΣA .. SA :: AB .. BC. R .. TA :: AB .. BC. τA .. R :: AB .. BC.	AC AB BC	1
AB A and C	AC	ΣA .. R :: AB .. AC. R .. ſA :: AB .. AC. τA .. σA :: AB .. AC.	AC AB BC	2
AB BC	A and C	AB .. BC :: R .. TA. Complement is C. BC .. AB :: R .. TC. Complement is A.	AB BC	3
AB BC	AC	AB .. BC :: R .. TA; Then. ΣA .. R :: AB .. AC, or $\surd$: ABq + BCq: = AC (*per* 47. 1 *Eucl. El.*)	AB AC	4
AB AC	A and C	AC .. AB :: R .. ΣA. AB .. AC :: R .. ſA.	AC AB	5
AB AC	BC	AC .. AB :: R .. ΣA; Then R .. TA :: AB .. BC, or $\surd$: ACq — ABq: = BC.	AC AB	6
AC A and C	AB	R .. ΣA :: AC .. AB. ſA .. R :: AC .. AB. σA .. τA :: AC .. AB.	AC AB BC	7

Axiom II.

In any Triangle the Sides are proportional to the Sines of the oppoſite Angles.

Demonstration.



Produce the lesser Side of the Triangle ABC, to wit AB to F, making AF = BC: Let fall the Perpendiculars BD, FE, upon the Side CA produc'd, if need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius BC = AF.

Now the Triangles ABD and AFE, having the ∠ A common to them both, and the ∠ D = ∠ E = ∟, are similar; wherefore (by 4. 6 *Eucl. Elem.*) AF (BC) ·· AB :: FE ·· BD; *viz.* :: SA ·· SC. *Q. E. D.*

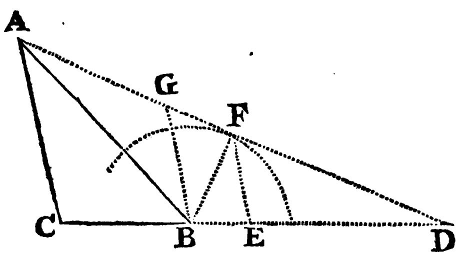
Otherwise thus;

By *Ax.* I. AB ·· R :: BD ·· SA, and BC ·· R :: BD ·· SC;
Therefore AB × SA (= R × BD) = BC × SC;
Wherefore AB ·· BC :: SC ·· SA. *Q. E. D.*

Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle, Is to their Difference, As the Tangent of half the Sum of the Angles opposite to those Legs, Is to the Tangent of half their Difference.

Demonstration.



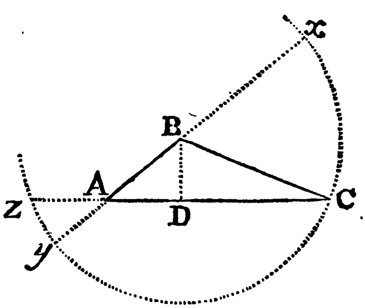
In the Triangle ABC produce CB, the lesser Leg of the Angle B, till BD becomes = BA, the greater Leg, and then bisect CD in E; join AD, and bisect it also in F; draw BF, which (by 8. 1 *Eucl. Elem.*) will be ⊥ AD; and draw EF,

EF, which (by 2. 6 *Eucl. Elem.*) will be || AC. Then will the Angle ABF = FBD = ½ ABD, which external Angle ABD is (by 32. 1 *Eucl. Elem.*) = BAC + C, that is to the Sum of the opposite Angles required.

Draw then BG parallel to CA, so will the Angle GBA be (by 29. 1 *Eucl. Elem.*) equal to its alternate one BAC; and if from half the Sum of the opposite Angles you take the lesser Angle, *i. e.* If from ∠ ABF you take the ∠ GBA, there will remain ∠ GBF = half the Difference of the opposite Angles: And so also, if from CE half the Sum of the Legs, you take CB the lesser Leg, there will remain BE equal to half the Difference of the Legs. And then, since the △ ABF is Right-angled, if BF be made Radius, AF will be the Tangent of ∠ ABF (*i. e.* the Tangent of half the Sum of the opposite Angles); and in the little △ GBF, FG will be the Tangent of the ∠ GBF (*i. e.* the Tangent of half the Difference of the opposite Angles): But the Segments of the Legs of any Triangle cut by Lines parallel to the Base, being (by *Schol.* to 2. 6 *Eucl. El.*) proportional; EC ·· EB :: FA ·· FG; that is in Words, half the Sum of the Legs, Is to half their Difference, As the Tangent of half the Sum of the opposite Angles, Is to the Tangent of half their Difference: But Wholes are as their Halves; wherefore the Sum of the Legs, Is to their Difference, As the Tangent of half the Sum of the Angles opposite, Is to the Tangent of half their Difference. *Q. E D.*

Axiom IV.

The Base, or greatest Side of any Plane Triangle, Is to the Sum of the Legs, As the Difference of the Legs, Is to the Difference of the Segments of the Base made by a Perpendicular let fall from the Angle opposite to the Base.



Demonstration.

From the ∠ B, on the Base AC, of the △ ABC, let fall the Perpendicular BD; on B, as a Center, with the greater Leg BC, as a Radius, describe the Circle B*x*C*y*Z; and produce AB to *x* and *y*, and CA to Z. Then,

By the 35. 3 *Eucl. Elem.* A y x A x is $=$ A C x A Z; viz. : B C $-$ B A : x : B C $+$ B A : $=$ A C x : D C $-$ D A :

Therefore A C ·· B C $+$ B A :: B C $-$ B A ·· D C $-$ D A. *Q. E. D.*

The Proportions for the Solution of the six Cases of Plane oblique Triangles. [*See the last* Fig.]

Given.	*Reqd.*	*Proportions.*	*Ax.*	*Case.*
A B B C and C	A	A B ·· B C :: S C ·· S A.	2	1
A B B C and C	A C	A B ·· B C :: S C ·· S A. Hence, by Subtraction, the ∠ B will be known. S A ·· S B :: B C ·· A C.	2	2
A, C and B C	A B	S A ·· S C :: B C ·· A B.	2	3
B A B B C	A and C	B C $+$ A B ·· B C $-$ A B :: T $\frac{1}{2}$ Sum of the ∠s opposite ·· T $\frac{1}{2}$ Difference of the ∠s opposite. Then $\frac{1}{2}$ Sum $+$ $\frac{1}{2}$ Difference $=$ greater ∠ A; and $\frac{1}{2}$ Sum $-$ $\frac{1}{2}$ Difference $=$ lesser ∠ C.	3	4
B A B B C	A C	First, find the Angles by the last; then S C ·· S B :: A B ·· A C.	3 2	5
A B B C A C	A B C	A C ·· B C $+$ B A :: B C $-$ B A ·· D C $-$ D A : Then $\frac{1}{2}$ A C $+$ $\frac{1}{2}$ D C $-$ $\frac{1}{2}$ D A $=$ D C. And $\frac{1}{2}$ A C $-$: $\frac{1}{2}$ D C $-$ $\frac{1}{2}$ D A : $=$ D A. Then A B ·· A D :: R ·· Σ A. And C B ·· D C :: R ·· Σ C. And 180° $-$ ∠ A $-$ ∠ C $=$ ∠ B.	4 1 1	6

PART

PART II.

The Solution of ſome Problems in Plane Geometry.

WHen you have a Geometrical Queſtion, or *Problem*, propos'd to be reſolv'd by ALGEBRA, the firſt Thing you are to do is to illuſtrate it by a Conſtruction, or an exact Deſcription, if you can do it; but, if you can't, deſcribe a Figure as nearly repreſenting the true one, and as fully ſetting forth the ſeveral Parts of the *Problem*, as you can gueſs; and ſuppoſe this *Figure* to be the true one, in which, ſome of the Lines being known or given, and ſome of them unknown, the ſame *Problem* may therefore be ſolv'd after various Ways or Manners. Next, having conſider'd the Nature of the *Problem*, you are thence to take a curſory View of the moſt obvious Ways of ſolving it; and, having choſen the beſt in your Opinion, then deſign the unknown Line or Lines you have choſen along with the given Lines or Quantities (or as many of them as are ſufficient to determine the *Problem*, and moſt fit for your Purpoſe) in proper *Symbols*. Afterwards, if it be requiſite, prepare the *Figure* by drawing and producing perpendicular, parallel, ſtraight, *&c.* Lines in ſuch Parts, and after ſuch Manners, in and about it, as are ſuitable to the Nature of the *Problem*, according to the Method of Solution you have before choſen, in order, by the Help of theſe Lines, or Mediums, to deduce a Connexion in the Operation, between the Lines and Quantities in the *Figure* deſign'd by the *Symbols*: And then proceed to the Operation, wherein the foregoing Rules and Methods, in this Treatiſe (with a competent Knowledge in *Euclid*'s *Elements*) will be your Guide.

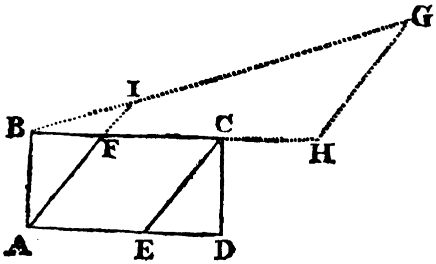
Now, as the moſt ſimple reduc'd Equation is beſt, you are always to endeavour to attain to it in the eaſieſt Manner poſſible: But, as it can't be reaſonably expected that you can ſucceed in this, in all Caſes, at the firſt, you may repeat your Endeavours in another Solution of the ſame *Problem*; and ſo on, till you hit upon the beſt. Theſe Repetitions will, in moſt Caſes, be needleſs to you after uſing yourſelf, with due Application,

Application, to the Solutions of various Sorts of *Problems*, beginning with some of the easiest, and proceeding gradually to more difficult ones, which (I believe) is the best Rule that can be given to a young *Mathematician*, in order to enable him to make a Progress in this most noble Science. I will therefore now proceed to such *Problems*.

CHAP. I.

Of Problems producing Simple Equations.

PROB. I.



TO inscribe a *Rhombus* in a given long Square, *i. e.* Having the Sides of the ⌊d ▭, *viz.* AB and BC given. To find the Segment BF or DE, which being cut off, the Remainder FC or AE will be the Side of the *Rhombus* sought.

Solution.

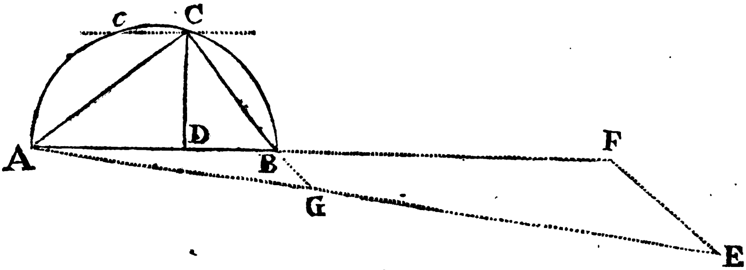
Let	1	$AB = b\ (= DC) = 20.$
	2	$BC = c\ (= AD) = 40.$
Suppose	3	$BF = x\ (= DE) = ?$
Then	4	$FC = c - x\ (= BC - BF) = FA = EC = EA.$
47. 1 *Euc. El.*	5	$bb + xx = FAq = cc - 2cx + xx.$
5,	6	$2cx = cc - bb.$
$6 \div 2c$	7	$x = \frac{1}{2}c - \frac{bb}{2c} = 15.$
Consequently	8	$2c \cdot\cdot c + b :: c - b \cdot\cdot x.$

Construction.

By 12. 6 *Eucl. El.* Make $BG = 2c \cdot\cdot BH = c + b :: BI = c - b \cdot\cdot BF = x = DE$. Then draw the straight Lines AF and CE, and you will have the *Rhombus* AFCE inscribed in the given ⌊d ▭ ABCD, as was required.

PROB.

PROB. II.



The *Perimeter* (*viz.* the Sum of the Sides A B, A C and B C) of any ⌞d △ A B C, and the Perpendicular C D let fall from the Right-angle C to the opposite Side A B, being given: Thence to find the Triangle.

Solution.

Let	1	$AB + BC + AC = b = 120.$
	2	$CD = c = 24.$
Suppose	3	$AB = x = ?$
And	4	$BC = y.$
Then	5	$AC = b - x - y.$
47. 1 *Eucl. Elem.*	6	$xx\ (= ABq) = bb - 2bx + xx - 2by + 2xy + 2yy\ (= ACq + BCq).$

The ⌞d △s A B C and C B D are similar; consequently,

4. 6 *Eucl. El.*	7	$x(AB) \cdot\cdot b - x - y(AC) :: y(BC) \cdot\cdot c(DC).$
$\because$	8	$cx = by - xy - yy.$
8×2	9	$2cx = 2by - 2xy - 2yy.$
$6 + 9$	10	$xx + 2cx = bb - 2bx + xx.$
10,	11	$2bx + 2cx = bb.$
$11 \div \overline{2b + 2c}$	12	$x = \dfrac{bb}{2b + 2c} = \dfrac{\frac{1}{2}bb}{b + c} = 50.$
Consequently	13	$b + c \cdot\cdot b :: \frac{1}{2}b \cdot\cdot x.$

Construction.

First, By 12. 6 *Eucl. El.* Make, As $AE = b + c \cdot\cdot AF = b :: AG = \frac{1}{2}b \cdot\cdot AB = x$. *Then on AB, as a Diameter, describe the Semi-circle A c C B. At the Distance

---

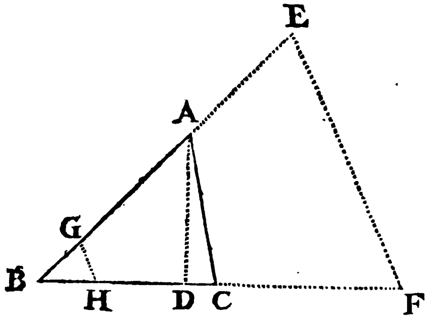
* N. B. *Thus far the* Problem *is Simple; but in the following Part thereof (a Circle being necessarily us'd in the Construction) it is a Plane* Problem.

2

$DC = c$

DC = c draw cC $\parallel$ AB, which $\parallel$ will intersect the Semicircle in the Points c and C. From either of the Points of Intersection C let fall CD $\perp$ AB. Draw the Lines AC and BC, and the Triangle is constructed.

PROB. III.



In an Acute-angled Triangle, having the three Sides given severally; to determine the Point where a Line perpendicularly let fall from the Vertex shall cut the Base; *i. e.* Having AB, AC, BC severally given; to find BD in the annex'd *Fig.*

Solution.

Suppose it done, and the Line (or $\perp$) AD drawn; then

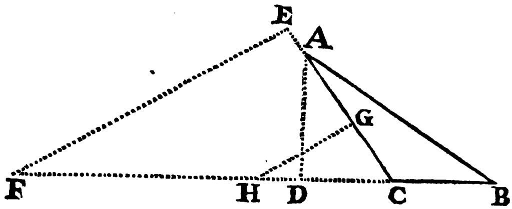
Put	1	$BC = b = 9.$
	2	$CA = c = 8.$
	3	$AB = d = 11.$
And suppose	4	$BD = a = ?$
Then	5	$DC = b - a.$
47. 1 *Eucl. El.*	6	$dd - aa = ADq = cc - bb + 2ba - aa.$
6,	7	$dd - cc + bb = 2ba.$
$7 \div 2b$	8	$\frac{dd - cc}{2b} + \frac{1}{2}b = a = 7\frac{2}{3}.$
Consequently	9	$2b \cdot\cdot d + c :: d - c \cdot\cdot \frac{dd - cc}{2b}.$
		And $\frac{dd - cc}{2b} + \frac{1}{2}b = a.$

Construction.

First, By 12. 6 *Eucl. El.* Make $BE = 2b \cdot\cdot BF = d + c :: BG = d - c \cdot\cdot BH$. Then $BH + \frac{1}{2}BC = a = BD$ sought.

PROB.

PROB. IV.



In an Obtuse-angled Triangle, having the three Sides given severally; to determine the Point where a Line let fall perpendicularly from the Vertex shall cut the Base produc'd, *i. e.* Having given AB, BC, AC, severally in the annex'd *Fig.* to find CD.

Suppose it done, and AD drawn; then

Solution.

Suppose	1	$BC = b = 10.$
	2	$AC = c = 16.$
	3	$AB = d = 24.$
And	4	$DC = a = ?$
Then	5	$BD = a + b.$
47. 1 *Eucl. El.*	6	$dd - aa - 2ab - bb = DAq = cc - aa.$
6,	7	$dd - cc - bb = 2ab.$
$7 \div 2b$	8	$\frac{dd - cc}{2b} - \frac{1}{2}b = a = 11.$
Consequently	9	$2b \cdot\cdot d + c :: d - c \cdot\cdot \frac{dd - cc}{2b}.$
		And $\frac{dd - cc}{2b} - \frac{1}{2}b = a.$

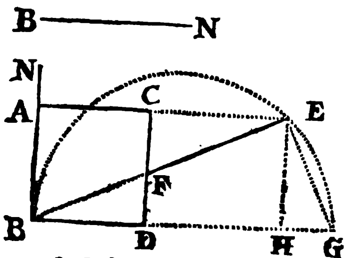
Construction.

First, By 12. 6 *Eucl. El.* Make $CE = 2b \cdot\cdot CF = d + c :: CG = d - c \cdot\cdot CH$. Then $HC - \frac{1}{2}BC = a = DC$ the Segment sought.

N. B. The $\triangle$s ACB in these two *Problems* being describ'd (by 22. 1 *Eucl. Elem.*) the Segments sought may be Geometrically determin'd only by letting fall the $\perp$ AD on the Base BC produc'd, if need be.

Note likewise, There are other Methods of constructing the foregoing *Problems*; but I don't think it worth while to insert any more of them.

A Problem, producing a Simple, Quadratick Equation.



HAving given the Square AD, and a straight Line BN, you are to produce the Side AC to E; so that EF drawn from E towards B shall be equal to BN.

It will be evident, if you imagine a Semi-circle to pass thro' the Points B and E, that the most commodious Way will be to find the Line DG, that you may have the Diameter BG, upon which, having afterwards describ'd a Semi-circle, there will be need of no other Operation to satisfy the *Problem*, than to produce the Side AC, till it meets the prescrib'd *Periphery*.

Solution.

Make	1	$BD = b = DC = 3.$
	2	$NB = c = FE = 4.$
Suppose	3	$DG = x = ?$
And	4	$BF = y.$
Then	5	$BG = b + x.$
And	6	$BE = y + c.$

The $\triangle$s BFD, GEH, and BEG are similar; wherefore

	7	$b\,(BD) \cdot\cdot\, y\,(BF) :: b\,(EH) \cdot\cdot\, y\,(EG).$
	8	$b+x\,(BG) \cdot\cdot\, y\,(EG) :: y+c\,(BE) \cdot\cdot\, b\,(EH)$
$\therefore$	9	$bb + bx = yy + yc.$
47. 1 *Eucl. El.*	10	$bb + 2bx + xx\,(BGq) = yy\,(EGq) + yy + 2cy + cc\,(BEq).$
10 — 9.	11	$bx + xx = yy + cy + cc.$
11 — cc.	12	$bx + xx - cc = yy + cy.$
12 = 9.	13	$bx + xx - cc\,(= yy + cy) = bb + bx.$
13,	14	$xx = bb + cc.$
14 *w.* 2.	15	$x = \sqrt{:} bb + cc : = 5.$

Construction.

Having produc'd the Side of the Square BA to N, so that BN shall be equal to the given straight Line BN; then, since BD is $= b$, and BN $= c$, the Hypothenuse DN will be $= \sqrt{:}\, bb + cc : = x$. Having therefore made DG $=$ DN, and

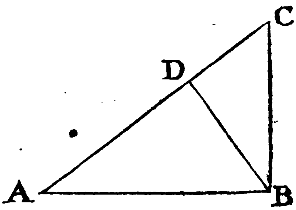
and describ'd a Semi-circle upon the whole Line BG, if AC be prolong'd until it meet the *Periphery* in E, you'll have done that which was required.

CHAP. II.

Problems producing affected Quadratick Equations.

PROB. I.

IN the ∟d. △ ABC, the Perpendicular BC, and the alternate Segment of the Hypothenuse (made by a Perpendicular let fall from the Right-angle) *viz.* AD being given; to find the other Segment DC.

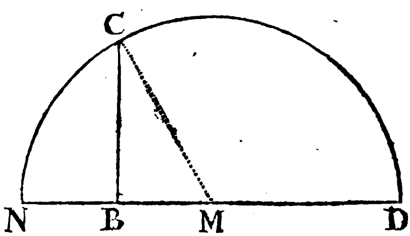


Solution.

Let	1	$BC = p =$ the $\perp = 45.$
	2	$b = AD = 48.$
Suppose	3	$a = DC = ?$
4. 6 *Eucl. El.*	4	$b \cdot\cdot BD :: BD \cdot\cdot a.$
$\because$	5	$ba = BDq.$
47. 1 *Eucl. El.*	6	$pp - aa = BDq.$
$5 = 6$	7	$ba = pp - aa.$
$7 + aa$	8	$aa + ba = pp.$ Case 1. of adf. Quadrat.
Comp. □.	9	$aa + ba + \frac{1}{4}bb = pp + \frac{1}{4}bb.$
$9\ \omega\ 2.$	10	$a + \frac{1}{2}b = \sqrt{:pp + \frac{1}{4}bb:}$
$10 - \frac{1}{2}b$	11	$a = \sqrt{:pp + \frac{1}{4}bb:} - \frac{1}{2}b = 27$ or $-75.$

Construction.

Make $BM = \frac{1}{2}b$, and erect the $\perp$ BC, which make $= p$. On M as a Center with the Radius MC, describe a Semi-circle NCL, intersecting the straight Line BM produc'd both ways in the Points N and L.



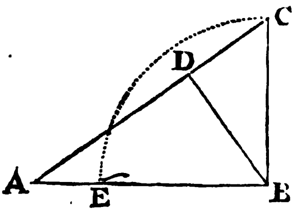
I say,

I say, that the straight Line NB is the affirmative Root, and LB the negative, of the Equation $aa + ba = pp$.

For seeing BM is half the Difference of the straight Lines NB and BL; if NB be put for a, then BL will be $a + b$; and therefore, since NB .. BC :: BC .. BL (*per* 13. 6 *Eucl. El.*), i. e. $a \,..\, p :: p \,..\, a + b$, $aa + ba = pp$, as before.

In like Manner, if $-$LB be $= a$, and consequently LB $= -a$, NB will be $-a - b$; And (by 13. 6 *Eucl. El.*) NB $\times$ LB is $=$ BCq; that is, $aa + ba = pp$: Wherefore the Construction is right.

PROB. II.

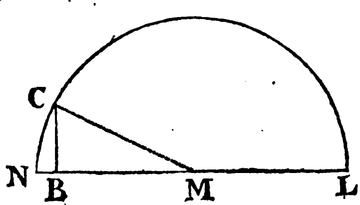


The Difference AE between the Base AB and Perpendicular BC of the ⌞d △ BAC, and the Perpendicular BD, let fall from the Right-angle ABC, upon the Hypothenuse AC being given; thence to find the Hypothenuse AC.

Solution.

Let	1	$d = AE = 15.$
And	2	$p = BD = 36.$
Suppose	3	$a = AC = ?$
And	4	$e = EB.$
4. 6 *Eucl. El.*	5	$d + e(AB) \,..\, p(BD) :: a(AC) \,..\, e = EB.$
∴	6	$de + ee = pa.$
47. 1 *Eucl. El.*	7	$dd + 2de + ee + ee (= ABq + BCq)$ $= aa = dd + 2de + 2ee.$
$6 \times \overline{2}.$	8	$2de + 2ee = 2pa.$
$7 - 8.$	9	$dd = aa - 2pa.$ Case 2.
$9 + pp.$	10	$dd + pp = aa - 2pa + pp.$
10 *w.* 2.	11	$\sqrt{:dd + pp:} = a - p.$
$11 + p.$	12	$\sqrt{:dd + pp:} + p = a = 75$ or $-3.$

Construction.



Make BM $= p$, and erect the ⊥ BC, which make $= d$. On M, as a Center, with the Radius MC describe a Semi-circle NCL intersecting the straight Line BM, produc'd both Ways in the

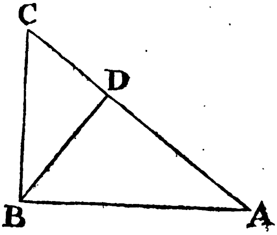
Points N and L. I ſay, that the ſtraight Line L B is the affirmative, and N B the negative Root of the Equation $aa - 2pa = dd$.

For ſeeing BM is half the Difference of the ſtraight Lines L B and B N, if L B be put for a, then B N will be $a - 2p$; And (by 13. 6 *Eucl. Elem.*) $LB \times BN = BCq$; that is, $aa - 2pa = dd$, as above.

In like Manner, if N B be $= -a$, then $LB = 2p - a$, and $NB \times LB = BCq$ is $-2pa + aa = dd$; wherefore the Conſtruction is right.

PROB. III.

The Hypothenuſe A C of any ∟d △ A B C, and the Perpendicular B D, let fall from the ∟ A B C upon the Hypothenuſe A C, being given; to find A D, the greater Segment of the Hypothenuſe.

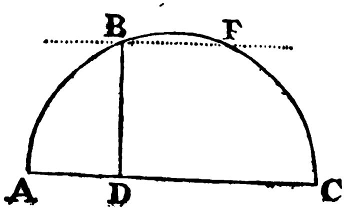


Solution.

Let	1	$h = AC = 75.$
	2	$p = BD = 36.$
Suppoſe	3	$AD = a = ?$
4. 6 *Eucl. El.*	4	$a \cdot\cdot p :: p \cdot\cdot DC.$
∴	5	$\frac{pp}{a} = DC.$
$1 - a.$	6	$h - a = DC.$
$5 = 6.$	7	$\frac{pp}{a} = h - a.$
$7 \times a.$	8	$pp = ha - aa.$ Caſe 3.
8,	9	$aa - ha = -pp.$
$9 + \frac{hh}{4}$	10	$aa - ha + \frac{1}{4}hh = \frac{1}{4}hh - pp.$
10 ɯ. 2.	11	$a - \frac{1}{2}h = \pm\sqrt{:\frac{1}{4}hh - pp:}$
$11 + \frac{1}{2}h.$	12	$a = \frac{1}{2}h \pm \sqrt{:\frac{1}{4}hh - pp:} = 48$, or 27; but, becauſe of the Limitation in the *Problem*, $a = 48.$

Conſtruction.

Construction.



Describe a Semi-circle, whose Diameter A C let be equal to h. Draw F B parallel to A C, at the Distance D B $=p$; which Parallel, if the Equation be possible, will intersect the Circle in the Points B and F; from the Point of Intersection B let fall the ⊥ B D to the Diameter A C. I say, that both A D and D C are affirmative Roots of the Equation $aa-ha=-pp$.

For A C or h being their Sum [*See Pag,*], if A D be put $=a$, D C will be $=h-a$; or if D C be $=a$, A D will be $h-a$; whence, in both Cases, $ha-aa$, or the Rectangle of A D × D C, will be equal to pp, or the Square of D B (*per* 13. 6 *Eucl. Elem.*), which was to be done.

This Equation sometimes becomes impossible, *viz.* when p is so great, as that the ∥ B F does neither cut nor touch the Circle A B C; that is, when p is greater than $\frac{1}{2}h$: For p ought to be a Geometrical Mean Proportional between the Parts (A D and D C) of h, and consequently not greater than an Arithmetical Mean, or $\frac{1}{2}h$; Nor are they equal, except in the Case of Contact; where likewise a and p become equal.

All simple Equations may be constructed by straight Lines only. [*See the four first* Problems *of this* Part II.]

But Plane, or Quadratick Equations require, besides straight Lines, the Circle, or some other Curve of the Conick-Sections to construct them. [*See the four last* Problems.]

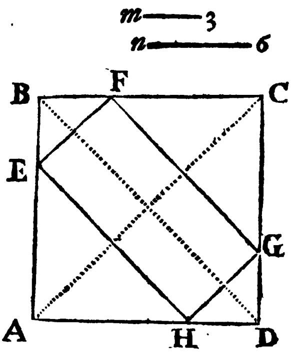
In the three last *Problems* you have Methods of constructing all original *adfected Quadratick Equations.* And as for those of higher Degrees, I will not insert here the Methods of constructing them; for, whenever 'tis required that any Thing in *Geometry* should be accurately determin'd, a *Mathematician* must not undertake to do it by Rule and Compass, because of the Defect of Instruments, and of our Senses, whereby the Intersections of Lines imperfectly drawn are yet more imperfect: But he will give a Solution as near the Truth as you please by an Arithmetical Calculation, according to an Equation determining the Nature of the *Problem.* Moreover, Cubick and Biquadratick Equations are not to be constructed by straight Lines and Circles; 'tis true, they may, by the additional Help of a *Parabola,* which, indeed, from the

Nature of its Equation, is more ſimple than the Circle; and higher Equations may be conſtructed by the Help of more compounded Curves; beſides, a Geometrical Conſtruction, rightly manag'd, ſhews at once, as well the Number and, nearly, Quantity of the real Roots, as their Signs, *viz.* whether they be affirmative or negative: But ſeeing the *Parabola* and more compounded Curves cannot be deſcribed but by Points, and the uncertain Motion of the Hand, the Antients hardly admitted them into their *Geometry*, and would ſcarce allow that to be Geometrically effected, which could not be deſcribed by the Help of the Rule and Compaſſes.

CHAP. III.

The Solution of Problems of ſeveral Sorts.

PROB. I.



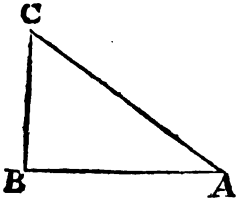
Within a given Square ABCD, to inſcribe a long Square EFGH, whoſe Sides FE (or GH) and FG (or EH) ſhall be parallel to the Diagonals AC and DB reſpectively, and in Proportion to each other As *m* To *n*.

Solution.

Let	1	$b = AB = AD = 12.$
Suppoſe	2	$a = BF = BE = HD = DG.$
Then	3	$b - a = FC = EA.$
47. 1 *Eucl. El.* {	4	$\sqrt{2aa}(=\sqrt{:EBq+BFq:}) = EF = a\sqrt{2}.$
	5	$:b-a:\times\sqrt{2}(=\sqrt{:FCq+CGq:}) = FG.$
By the *Problem*	6	$m(3) \cdot\cdot n(6) :: a\sqrt{2} \cdot\cdot \overline{b-a}\sqrt{2}.$
$\because$	7	$m \cdot\cdot n :: a \cdot\cdot b - a.$
$\because$	8	$mb - ma = na.$
$8 + ma.$	9	$mb = na + ma.$
$9 \div \overline{n+m}.$	10	$\frac{mb}{n+m} = a = 4.$

PROB.

PROB. II.



The *Perimeter* (*viz.* the Sum of the three Sides AB, AC, BC) of any ∟d △ ABC, and its Area being given; thence to find each Side.

Solution.

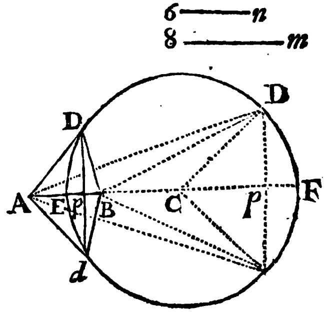
Let	1	$a = AC.$
	2	$e = AB.$
	3	$y = BC.$
Then	4	$a + e + y =$ *Perimeter* $= b = 12.$ } by the *Prob.*
	5	$\frac{1}{2} ey =$ the Area $= c = 6.$
47. 1 E. El.	6	$yy + ee = aa.$
$5 \times 4.$	7	$2ey = 4c.$
$6 + 7.$	8	$yy + 2ey + ee = aa + 4c.$
$4 - a.$	9	$e + y = b - a.$
$9 \odot 2.$	10	$ee + 2ey + yy = bb - 2ba + aa.$
$8 = 10.$	11	$aa + 4c = bb - 2ba + aa.$
11,	12	$2ba = bb - 4c.$
$12 \div 2b.$	13	$a = \frac{1}{2}b - \frac{2c}{b} = 5.$
9, 13.	14	$e + y = \frac{1}{2}b + \frac{2c}{b}.$
$6 - 7.$	15	$ee - 2ey + yy = aa - 4c.$
15, 13.	16	$ee - 2ey + yy = \frac{1}{4}bb - 6c + \frac{4cc}{bb}$
16 ɯ. 2.	17	$e - y = \pm\sqrt{:\frac{1}{4}bb - 6c + \frac{4cc}{bb}:}$
$14 + 17.$	18	$2e = \frac{1}{2}b + \frac{2c}{b} \pm \sqrt{:\frac{1}{4}bb - 6c + \frac{4cc}{bb}:}$
$18 \div 2.$	19	$e = \frac{1}{4}b + \frac{c}{b} \pm \sqrt{:\frac{1}{16}bb - \frac{1}{2}c + \frac{cc}{bb}:} = \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}.$
$14 - 17.$	20	$2y = \frac{1}{2}b + \frac{2c}{b} \mp \sqrt{:\frac{1}{4}bb - 6c + \frac{4cc}{bb}:}$
$20 \div 2.$	21	$y = \frac{1}{4}b + \frac{c}{b} \mp \sqrt{:\frac{1}{16}bb - \frac{1}{2}c + \frac{cc}{bb}:} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}.$

2

PROB. III.

(Apol. Perg.)

Two Points A, B in a Plane being given; and, knowing it poſſible to be done; ſo to deſcribe in the ſame Plane a Circle D, D, that to every Point of its Circumference D, the ſtraight Lines AD, BD being drawn, ſhall be in the given Ratio of m the greater to n the leſſer.



Inquiſition.

Suppoſe it done, and C the Center of the Circle: Then, becauſe the Points A, B are given, the Line AB (which ſuppoſe $= 7$) is given; and becauſe of m to n, its Point E, this being one of the Points D; therefore AE and BE are given.

Put AE $= b(4)$; then $m(8) \cdot\cdot\, n(6) :: b(4) \cdot\cdot\, \frac{n}{m}b(3) =$ BE.

Suppoſing then from D on AB produc'd, if need be, a $\perp$ Dp drawn, and taking (on oppoſite Sides of AB) AD $=$ Ad, and BD $=$ Bd, the Δs on oppoſite Sides of AB will be ſimilar and equal; and therefore D$p$$d$ one ſtraight Line biſected in p, and at $\llcorner$s to AB in which therefore is the Center C, and the Diameter ECF, which muſt lie towards the Side B, not A, otherwiſe BF would be $\sqsubset$ AF, that is here BD $\sqsubset$ AD.

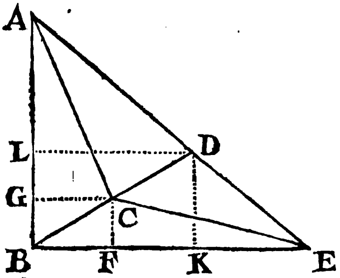
Now ſuppoſe EC $= r = ?$ And Ep $= x$; then Ap $= b + x$, and Bp $= \frac{n}{m} b \backsim x$; then (by the Property of a Circle) $2rx - xx = p\mathrm{D}q$: And (by 47. 1 *Eucl. Elem.*) $2rx - xx + bb + 2bx + xx = \mathrm{AD}q = 2rx + 2bx + bb$: As alſo $2rx - xx + \frac{nn}{mm}bb - \frac{2n}{m}bx + xx = \mathrm{BD}q = 2rx - \frac{2n}{m}bx + \frac{nn}{mm}bb$.

Now (by *Hypotheſis*) $m \cdot\cdot\, n ::$ AD $\cdot\cdot$ BD; Conſequently, $mm \cdot\cdot\, nn :: \mathrm{AD}q \cdot\cdot\, \mathrm{BD}q$; that is $:: 2rx + 2bx + bb \cdot\cdot\, 2rx - \frac{2n}{m}bx + \frac{nn}{mm}bb$; wherefore $2mmrx - 2nmbx +$

$nnbb = 2nnrx + 2nnbx + nnbb \therefore mmr - nnr = mnb + nnb$; Consequently, by dividing each Part by $mm - nn$ you'll have $r = \frac{nb}{m-n} \left(= \frac{6 \times 4}{8-6} = 12.\right)$

If m be $= n$, then r will be infinite; and (since by how much the greater the Radius of a Circle is, by so much the nearer its Circumference approaches to a straight Line) the Circumference d E D will become an endless straight Line, perpendicular to A B, and bisecting it in E.

PROB. IV.



The Angle A B E being Right, the ∠ C B E and Side C B being given, as also C D = C B; It is required to draw the straight Line A D E thro' the Extremity of the given Line B C D, so as the Lines drawn from C to the Angles A and E, may be equal, *viz.* A C = C E.

Analysis.

1. Suppose it done, and through C draw C G ∥ E B, and C F ∥ A B.
2. Thro' D likewise draw D L ∥ E B, and D K ∥ A B.
3. In the △ B C F you have the ∠s B and F, and the Side C B given; And therefore the Sides B F and C F are easily found by Trigonometrical Calculation; wherefore, suppose them known; And put C F $= b$ $(= 5)$ and B F $= d$ $(= \sqrt{75}.)$
4. Put B C $= c$ $(= 10)$ and therefore B D $= 2c$. Then

4. 6 *E. El.*	5	L D = B K $= 2d$, and D K = L B $= 2b$.
Suppose	6	K E $= a = ?$
∴	7	F E $= d + a$.
47. 1 *E. El.*	8	□ C E = □ F E + □ C F $= dd + 2da + aa + bb$.
4. 6 *E. El.*	9	a(KE) ⋅⋅ $2b$(KD) :: $2d$(LD) ⋅⋅ $\frac{4db}{a}$ = LA.
	10	AG = AL + LG $= \frac{4db}{a} + b = \frac{4db + ba}{a}$.
47. 1 *E. El.*	11	□ A C = □ A G + □ G C $= \frac{16ddbb + 8dbba + bbaa}{aa} + dd$.

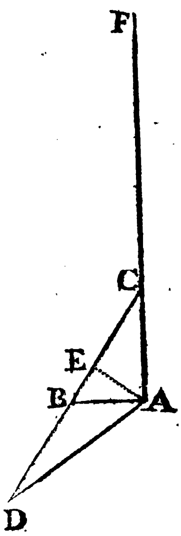
Now,

Now, by Suppoſition, the Square-Root of the 8th Step is = the Square-Root of the 11th ∵ the 8th Step is = the 11th.

Viz.	12	$dd + 2da + aa + bb = \dfrac{16ddbb + 8db^2a + bbaa}{aa} + dd.$
12,	13	$a^4 + 2da^3 - 8dbba - 16ddbb = 0.$
$13 \div \overline{a + 2d}$	14	$a^3 - 8dbb = 0$; conſeq. $a = \sqrt[3]{8ddbb}$ $(= \sqrt[3]{200\sqrt{75}} = 12 \cdot 00 +.)$

PROB. V.

A Tree AF b (200) Foot high, ſtanding upon the Side of a Hill, was by a tempeſtuous Wind broke in a Point C. The upper Part of it CF fell ſo as to become CD: And the Diſtance from D its Top, to A its Root, was found to be c (95) Foot: And a horizontal Line AB being drawn, till it cut the Part CD of the Tree in B, was found to be d (40) Foot. It is required to tell how many Foot long the ſtanding Part AC is.



In order to ſolve this *Problem*, I ſhall ſuppoſe the Tree AF to be perpendicular to the Horizon; And then, in the △ ACD, I have DC + CA = b, AD = c, Angle CAB Right, and AB = d given; to find CA = a = ?

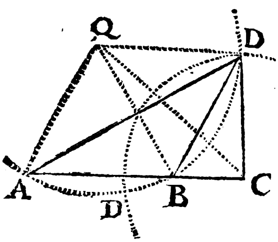
Let fall AE ⊥ DC, ſo is DC divided into two Segments DE and EC.

Solution.

Ax. IV. *of* Plane Trigon.	1	$b-a \cdot\cdot c+a :: c \backsim a \cdot\cdot \frac{cc \backsim aa}{b-a} = DE \backsim EC.$
	2	$\frac{b-a}{2} \mp \frac{cc \backsim aa}{2b-2a} \left(= \frac{b-a}{2} - \frac{cc-aa}{2b-2a}\right.$, or $\left.\frac{b-a}{2} + \frac{aa-cc}{2b-2a}\right) = \frac{bb-cc-2ba+2aa}{2b-2a} =$ $EC = \frac{f-ba+aa}{b-a}$ (putting $2f = bb-cc$)
47. 1 *E. El.*	3	$BC = \sqrt{: aa+dd :}$
4. 6 *E. El.*	4	$\sqrt{: aa+dd :} \cdot\cdot a :: a \cdot\cdot \frac{aa}{\sqrt{: aa+dd :}} = EC$
$2 = 4$	5	$\frac{f-baa+aa}{b-a} = \frac{aa}{\sqrt{: aa+dd :}}$
5 Reduc'd	6	$\begin{matrix}+2f \\ +dd\end{matrix} a^4 \begin{matrix}-2bf \\ -2bdd\end{matrix} a^3 \begin{matrix}+ff \\ -2ddf \\ +bbdd\end{matrix} a^2 - 2bddfa + ddff = 0.$

(The *6th* Step express'd in Numbers, and then divided by $\frac{25}{4}$, gives $5212\,a^4 - 1093600\,a^3 + 56547625\,aa - 1585920000\,a + 61404840000 = 0$. Now, if g be suppos'd $= 60$, a will be found, by Dr. *Halley*'s Rational *Theorem*, $= 60.54214$, and, by his Irrational *Theorem*, $a = 60.54217$.)

PROB. VI.



In the annex'd $\triangle$ are given
$AB = 2$
$BC = 1$; that is, $AC = 3$.
$CD = 1.7320508 +$
$\angle\, ADB = 30°$.
Required to find AD, *&c.*

Construction.

On $AB = 2$ make an *Isosceles* $\triangle\, AQB$, whose $\angle\, Q$ shall be $= 2 \angle\, ADB = 60°$. On Q, as a Center with the Radius QB or QA, describe the Circle ABD. Produce the Line AB to C, making $BC = 1$; then on C, as a Center with the Radius $CD = 1.732 +$ describe the Circle

DD

DD cutting the former ABD in the two Points D, D. Now $\left\{\begin{matrix}\text{ſince}\\ \text{if}\end{matrix}\right\}$ the given ∠ ADB $\left\{\begin{matrix}\text{is}\\ \text{was}\end{matrix}\right\}$ $\left\{\begin{matrix}\sqsupset\\ \sqsubset\end{matrix}\right\}$ ∟, from the $\left\{\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}\right\}$ D draw the ſtraight Lines DC, DB and DA, and the △ is conſtructed.

Method of Calculation.

Draw QD and QC.

1. The ∠ AQB being, by 20. 3 *Eucl. Elem.* = twice the given ∠ ADB is therefore given, which ſubtract from 180°; and half the Remainder is = ∠ QAB = ∠ QBA, by 32. and 5. of 1 *Eucl. Elem.*; wherefore, in the △ AQB you have all the Angles and the Side AB given, by which you may find, by *Ax.* II. *Pl. Trig.* the Side QA = QB.

2. From 180° ſubtract the ∠ QBA given, by the 1ſt Step, the Remainder is = ∠ QBC. Then, in the △ QBC, you have the ∠ B, the given Side BC and the Side QB given by the *1ſt* Step; by which you may find, by *Ax.* III. *Pl. Trig.* the Side QC, and the ∠s Q and C.

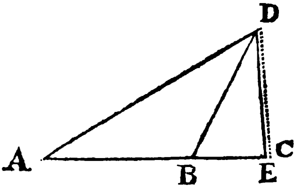
3. In the △ QDC you have the given Side DC, and the Sides QD (= QB, by the Definition of a Circle) and QC given, by the 1ſt and 2d Steps; whence you may find, by *Ax.* IV. *Pl. Trig.* the ∠s Q, C and D.

4. The Sum of the ∠s DQC and CQB given, by the 3d and 2d Steps, is = ∠ BQD = 2 ∠ DAB, by 20. 3 *Eucl. El.*; whence ∠ DAB is given: So in the △ DAC you have the ∠ A and the Sides AD and AC given, whereby you may find the Side required AD; &c.

5. Or, in the △ ADC you have the given Sides AC and CD, and the Sum of the ∠s QCB and QCD given by the 2d and 3d Steps; whence, &c.

An

An Algebraical Solution of PROB. VI.



$AB = b\ (=2)$ }
$BC = c\ (=1)$ } that is, $AC = b + c = f\ (=3)$ }
$CD = d\ (= 1.7320508 +$; that is, $CD = \sqrt{3}\,)$ } Given
$S \angle ADB = s\ (= S\ 30^{\circ} = 5$, the Radius being $10)$ }
Required to find AD.

From D let fall $DE \perp AC$ produc'd if need be; then

Suppose	1	$EC = a = ?$ then $BE = c \mp a$, or $= a - c$, and $AE = f \mp a$.
47. 1 *Euc. Elem.*	2	$\sqrt{:dd - aa:} = ED$.
	3	$\sqrt{:cc \mp 2ca + aa + dd - aa:} = BD = \sqrt{:cc + dd \mp 2ca:}$ { N. B. $\overline{c - a}\vert^2 = \overline{a - c}\vert^2$.
	4	$\sqrt{:ff + dd \mp 2fa:} = AD$.
Ax. II. *Pl. Trig.*	5	$b \,..\, s :: \sqrt{:ff + dd \mp 2fa:} \,..\, \frac{s}{b}\sqrt{:ff + dd \mp 2fa:} = S \angle ABD = S \angle DBC$.
Ax. I. *Pl. Trig.*	6	$r \,..\, \sqrt{:cc + dd \mp 2ca:} :: \frac{s}{b}\sqrt{:ff + dd \mp 2fa:} \,..\, \sqrt{:dd - aa:}$

That is, putting $cc + dd = k$, and $ff + dd = l$.

	7	$r \,..\, \sqrt{:k \mp 2ca:} :: \frac{s}{b}\sqrt{:l \mp 2fa:} \,..\, \sqrt{:dd - aa:}$
Conseq.	8	$rr \,..\, k \mp 2ca :: \frac{ssl \mp 2ssfa}{bb} \,..\, dd - aa$.
∴	9	$rrbbdd - rrbbaa = kssl \mp 2kssfa \mp 2cssla + 4cssfaa$.
9,	10	$\frac{rrbbdd - kssl}{rrbb + 4cssf} = aa \mp \frac{2kssf + 2cssl}{rrbb + 4cssf}a$.

Now, for the first Term of, or the known or absolute Quantity in, the last Step put m, and for the Co-efficient of

2 a,

a, in that Step put $\mp 2n$, then the ſaid Step will become

$$11 \text{ Reduc'd} \left| \begin{array}{l|l} 11 & m = aa \mp 2na. \\ 12 & \pm n \pm \surd : m + nn : = a. \end{array} \right.$$

That is to ſay, if the Perpendicular DE falls to the Left-hand of C, then $a = n \pm \surd : m + nn :$ Caſe 1.

If the Perpendicular DE falls to the Right-hand of C, then $a = -n \pm \surd : m + nn :$ Caſe 2.

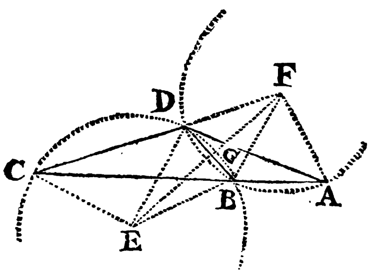
But if the ſaid Perpendicular coincides with DC, as it does according to the given Numbers between the foregoing Parentheſes, then either of the above *Theorems* will give one of the Values of $a = 0$.

(Thus, in *Caſe* 1. $a = \frac{6}{7} \pm \surd : 0 + \overline{\frac{6}{7}}\Big|^2 : = \frac{12}{7}$, or $= 0$.

The Reaſon of which is this; the given ∠ ADB being 30°, a then will be $= 0$, becauſe DE then coincides with DC: But if the ſaid Angle were 150°, as it may, (for 5, the Sine of 30°, is alſo the Sine of 150°.) then the ⊥ DE would be on the Left-hand of C; and therefore the other Value of a is $\frac{12}{7}$.)

The Value of a being thus had, the required Line AD may be found by the 4th Step ($= \surd : 9 + 3 - 0 :$ or $= \surd : 9 + 3 - 6 \times \frac{12}{7} :$, *viz.* $= \surd 12$, or $= \surd \frac{12}{7}$.)

Or EC $= a$ being found as before, the other Sides and Angles of the above △ or △s, and conſequently the Side required AD, &c. may be found by *Plane Trigonometry*.



PROB. VII.

In the annex'd △

$$\left.\begin{array}{l} \left.\begin{array}{l} AB = 1, \\ BC = 2, \end{array}\right\} \text{that is, } AC = 3. \\ \left.\begin{array}{l} \angle ADB = 30° \\ \angle BDC = 120° \end{array}\right\} \text{that is, } \angle ADC = 150° \end{array}\right\} \text{being Given.}$$

'Tis requir'd to find the other Sides and Angles.

Con-

Construction.

On BC $=2$ make an *Isosceles* △ BEC, whose ∠ E, or its Complement to $360°$ shall be $= 2$ ∠ BDC $= 240°$. On E as a Center, with the Radius EB or EC, describe the Circle BDC.

On AB $= 1$ make an *Isosceles* △ AFB, whose ∠ F shall be $= 2$ ∠ ADB $= 60°$. On F as a Center, with the Radius FA or FB, describe the Circle ABD intersecting the former Circle in the Points B and D. Draw AD, BD and CD, and the △ is constructed.

Method of Calculation.

Draw DE and DF and FE intersecting DB in G.

1. EB $=$ EC (by *Construction*;) therefore, by 5. 1 *Eucl. Elem.* ∠ EBC $=$ ∠ ECB will be given; for the Sum of them both is $=$ the Complement of the given ∠ BEC to $180°$: Wherefore, in the △ BEC, you have all the Angles, and the Side BC given; consequently you may, by *Ax.* II. *Plane Trig.* find BE $=$ EC. The same Way ∠ FAB $=$ ∠ FBA will be found, and AF $=$ FB.

2. If from ∠ EBA ($=$ here Sum of $180°$, and of the ∠ EBC found by 1.) you take the ∠ ABF (found by 1.) the Remainder ∠ FBE will be given.

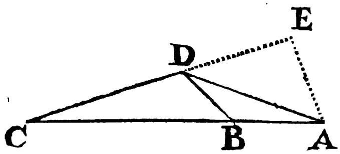
3. Now in the △ FBE you have the Sides FB and BE (*per* 1.) and ∠ FBE (*per* 2.) given, by which (*per Ax.* III. *Plane Trig.*) you may find the ∠ BFE.

4. The △s EDF and EBF have the Side EF common to both of them, and the Sides DF and DE equal to BF and BE respectively; therefore (by 8. 1 *Euclid's Elem.*) ∠ BFG $=$ DFG. Then the △s GBF and GDF have the Side FG common to them both, as also the Side FB $=$ FD, and the ∠ BFG $=$ DFG; wherefore, by 4. 1 *Eucl. Elem.* the ∠ BGF is $=$ ∠ DGF; *viz.* FG is ⊥ BD; and GD $=$ GB.

5. Now ∠ BFE being found (*per* 3.) and ∠ FGB $= 90°$, the ∠ GBF $=$ Complement of ∠ BFE to $90°$ will be given: Consequently, in the △ FGB you have all the ∠s and (*per* 1.) the Side FB given; whence you may (by *Ax.* I. *Plane Trig.*) find GB $=$ GD $= \frac{1}{2}$ DB: And then you are in a Way of finding the rest by *Plane Trigonometry*.

An

An Algebraical Solution of the foregoing Problem.



$$\left.\begin{array}{l}\text{Let } AB = b\ (=1)\\ \quad\ \ BC = c\ (=2)\end{array}\right\}\text{that is, } AC = b + c\ (=3)\ \left.\begin{array}{l}\\ \\ \end{array}\right\}\text{Given}$$

$$\left.\begin{array}{l}S\angle ADB = f\ (= {}^{*}5)\\ S\angle BDC = g\ (=\sqrt{75})\end{array}\right\}\text{and } \Sigma\angle ADC = m\ (=\sqrt{75})$$

And let $\angle$ ADC be $=$ L.

Required to find AD $= a = ?$

Solution.

Let fall AE $\perp$ CD produc'd.

Ax. II. *Pl. Trig.*	1	$b \,..\, f :: a \,..\, \frac{fa}{b} = S\angle ABD = S\angle DBC.$
	2	$g \,..\, c :: \frac{fa}{b} \,..\, \frac{cfa}{gb} = DC.$
12. 2 *Euc. Elem.*	3	$:\frac{bb+2bc+cc}{2} - \frac{aa}{2} - \frac{ccffaa}{2ggbb}: \div \frac{cfa}{gb} =$ $ED = \frac{b^3g+2bbcg+ccbg-aabg}{2cfa} - \frac{cfa}{2bg}.$
Ax. I. *Pl. Trig.*	4	$m \,..\, \frac{b^3g+2bbcg+ccbg-bgaa}{2cfa} - \frac{cfa}{2bg} :: r \,..\, a.$
∵	5	$ma = \frac{rb^3g+2rbbcg+rccbg-rbgaa}{2cfa} - \frac{rcfa}{2bg}.$
5 Reduc'd	6	$a = \frac{:b+c:\times bg}{\sqrt{:ccff+bbgg+\frac{2mbgcf}{r}:}}$

$(= \sqrt{\frac{27}{13}} = \sqrt{2.0769230769}$ &c. $= 1.44115+)$.

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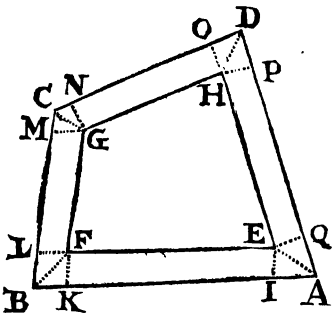
* N. B. *Radius here is* $= 10$.

PROB. VIII.



A Trapezium, *viz.* its Sides and Angles severally being given; To make another Trapezium within the former in such Manner, that the Sides of the one may be every where separated from the Sides of the other by an equal parallel Distance, and that the Area of the Space, lying between both Trapezia, may be equal to a given Space.

Let ABCD be the given Trapezium; and let the Sum of its Sides, *viz.* $AB + BC + CD + DA = b$:

As also EFGH the Trapezium to be inscribed; so as that EF be $\parallel$ AB, FG $\parallel$ BC, GH $\parallel$ CD, and HE $\parallel$ DA; and EI and FK being drawn $\perp$s AB, FL and GM $\perp$s BC, GN and HO $\perp$s CD, and HP and EQ $\perp$s DA, that the said $\perp$s may be equal to each other, and that the Area of the Space lying between both Trapezia be $= c$ a given Quantity.

Then 'tis plain, that the Sum of the Areas of the Trapezia IQ, KL, MN and OP, and of the $\llcorner$d $\square$s IF, LG, NH and PE is $= c$, and that $EF (= IK) = AB - AI - KB$, $FG (= LM) = BC - BL - MC$, $GH (= NO) = CD - CN - OD$, and $HE (= PQ) = DA - DP - QA$.

Solution.

Draw the Lines AE, BF, CG and DH; and suppose $EI = a = FK = FL =$ &c. $= ?$

In the $\llcorner$d $\triangle$s AEI and AEQ you have $AIq = AEq - EIq$, and $AQq = AEq - EQq\ (EIq)$, *per* 47. 1 *Euc. Elem.* consequently $AI = AQ$; wherefore the $\triangle$ AEI is $=$ AEQ; and the Angle $EAI = \frac{1}{2} QAI$ is given; and therefore the $\angle$ AEI $=$ Complement of $\angle$ EAI to a $\llcorner$ $(= \angle AEQ)$ is given, and consequently its Tangent, for which put f.

In like Manner the $\angle$ BFK $=$ BFL is also given, and let g be $=$ its Tangent.

Also the $\angle$ CGM $=$ CGN is likewise given, and put $h =$ its Tangent.

And

And the $\angle$ DHO $=$ $\angle$ DHP is given, whoſe Tangent deſign by k.

Then, by *Ax.* I. *Pl. Trigon.* r (Radius) .. a (EI) :: f .. $\frac{fa}{r}$ = AI = AQ:

Wherefore $\frac{faa}{2r}$ = Area of $\triangle$ AEI = Area of $\triangle$ AEQ;
that is to ſay, $\frac{faa}{r}$ = Area of the Trapezium IQ:

In like Manner will be found $\frac{gaa}{r}$ = Area of the Trapezium KL; alſo $\frac{baa}{r}$ = Area of the Trapezium MN;

And $\frac{kaa}{r}$ = Area of the Trapezium OP.

And : $b - \frac{2fa}{r} - \frac{2ga}{r} - \frac{2ba}{r} - \frac{2ka}{r} : \times a = ba - 2taa$ (putting $\frac{f+g+b+k}{r}$ equal to t) is = Sum of the Areas of the ⌞d ▭s IF, LG, NH and PE:

Conſequently, $taa + ba - 2taa = c$; which Equation, after due Reduction, gives $a = \frac{b}{2t} \pm \surd : \frac{bb}{4tt} - \frac{c}{t}$:

Note, In ſome Caſes c may be ſo great in Reſpect of the given Trapezium, as to reduce the inſcribed Trapezium to a $\triangle$ or |: And in other Caſes the above Value of a may be Real; and notwithſtanding neither a Trapezium $\triangle$ or | in ſuch Manner as is required can be inſcribed in the given Trapezium.

Examples in Numbers.

Let AB = 40	$\angle$ EAI = EAQ =	35°	15′	}	Given.
BC = 24	$\angle$ FBK = FBL =	44	06		
CD = 29	$\angle$ GCM = GCN =	55	56		
DA = 36	$\angle$ HDO = HDP =	44	43		
b = 129					

Conf. $\angle AEI =$	$\angle AEQ = 54^\circ\ 45'$	its Tangents	$*141496.73 = f$	
$\angle BFK =$	$BFL = 45\ \ 54$		$103191.99 = g$	
$\angle CGM =$	$CGN = 34\ \ 04$		$67620.28 = h$	
$\angle DHO =$	$DHP = 45\ \ 17$		$100993.94 = k$	
			$4.1330294 = t$	
			2	
			$8.2660588 = 2t.$	

* *Note*, Radius $= r$ is here $= 100000$.

And let c be $= 450.027$; then

$8.266)\ 129 = 15.60 = 2t)\ b \because 4tt)\ bb = 243.36$

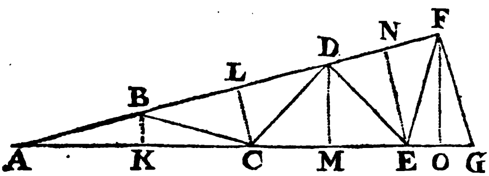
$4.13303)\ 450.27 = t)\ c = \quad 108.88$

$\quad 134.48$

$15.60 - \sqrt{134.48} = a = 15.60 - 11.59 = 4.00 +$;
That is, a is somewhat, but very little more, than 4.

PROB. IX.

To multiply or divide a **Given Angle** *by a* **Given Number.**



In the Given Angle FAG inscribe the Lines AB, BC, CD, DE, &c. of any Convenient, and the same Length, and ABC, BCD, CDE, DEF, &c. will be *Isosceles* $\triangle$s; and therefore, by 32. 1 *Eucl. El.* the Angle CBD will be $=$ $\angle$s A + ACB $= 2 \times \angle A$; and $\angle DCE = \angle$s A + ADC $= 3 \times \angle A$; and $\angle EDF = A + AED = 4 \times \angle A$; and $\angle FEG = 5 \times \angle A$; &c. Putting now AB, BC, CD, &c. the Radii of equal Circles, the Perpendiculars BK, CL, DM, &c. let fall on AC, BD, CE, &c. will be the Sines of the Angles BAC, CBD, DCE, &c. and AK, BL, CM, &c. the Sines of their Complements respectively.

Let therefore AB be $= r\ (10)$, AK $= c$ ($= \frac{1}{2}\sqrt{150} + \frac{1}{2}\sqrt{0}$ = Sine of 75°, the $\triangle$ A being 15°); then, by 4.

6 *Eucl. El.* $r\,(AB) \cdot\cdot c\,(AK) :: 2c\,(AC) \cdot\cdot \frac{2cc}{r} = AL$.

And $AL - AB = \frac{2cc}{r} - r = BL$ the Co-sine of $2 \times \angle A\,(=\surd 75)$. $r\,(AB) \cdot\cdot c\,(AK) :: \frac{4cc}{r} - r\,(= AD = 2AL - AB) \cdot\cdot \frac{4ccc}{rr} - c = AM$; And $AM - AC = \frac{4ccc}{rr} - c - 2c = \frac{4c^3}{rr} - 3c = CM$, the Co-sine of $3\angle A\,(=\surd 50.)$

$r \cdot\cdot c :: \frac{8c^3}{rr} - 4c\,(= AE = 2AM - AC) \cdot\cdot \frac{8c^4}{r^3} - \frac{4cc}{r} = AN$.

$AN - AD = \frac{8c^4}{r^3} - \frac{4cc}{r} - \frac{4cc}{r} + r = \frac{8c^4}{r^3} - \frac{8cc}{r} + r = DN$, the Σ of $4\angle A\,(=5)$.

$r \cdot\cdot c :: \frac{16c^4}{r^3} - \frac{12cc}{r} + r\,(= AF = 2AN - AD) \cdot\cdot \frac{16c^5}{r^4} - \frac{12c^3}{rr} + c = AO$: And $AO - AE = \frac{16c^5}{r^4} - \frac{12c^3}{rr} + c - \frac{8c^3}{rr} + 4c = \frac{16c^5}{r^4} - \frac{20c^3}{rr} + 5c = EO$, the Σ of $5\angle A\,(=\frac{1}{2}\surd 150 - \frac{1}{2}\surd 50)$: And so on.

Or, if you would have an Angle divided into any Number of equal Parts; putting $q = BL, CM, DN$, &c. severally, and $x = c$, you'll have $2xx - rr = rq$ an Equation, wherein x is equal to the Co-sine of half that Angle, whose Co-sine is q.

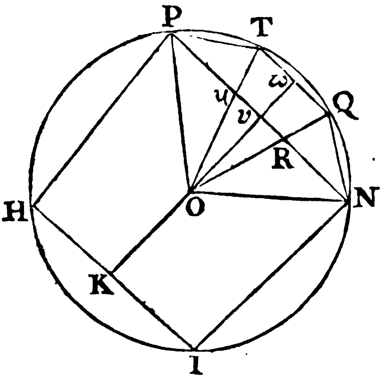
Also $4x^3 - 3rrx = qrr$, an Equation, wherein x is equal to the Co-sine of $\frac{1}{3}$ that Angle, whose Co-sine is q.

Also $8x^4 - 8r^2x^2 + r^4 = r^3q$, an Equation, wherein x is the Σ of $\frac{1}{4}$ that Angle, whose Σ is q.

Also

Also $16x^5 - 20r^2x^3 + 5r^4x = qr^4$, an Equation, wherein x is the Co-sine of $\frac{1}{5}$ of that Angle, whose Co-sine is $= q$, &c.

This Problem may be otherwise solv'd; a Specimen of the Method I will here insert in the Trisection of an Angle.



To divide a Given Angle N O P into three equal Parts; *i. e.* Having given the Chord of the Arc N P, and the Radius N O; to find N Q, the Subtense of the Third Part of the Given Arc.

Let N O be $= r$, N P $= b$, and N Q $= x = ?$

The $\triangle$s N O Q, QNR and R O y are similar; for the $\angle$ QON is (by Supposition) $=$ ROy (QOT) $= \frac{1}{2}$QOP, as also $\angle$ QNR (by 20. 3 *Eucl. Elem.*) is $= \frac{1}{2}$ QOP; Therefore the Angles QON, QNR and ROy are equal; also the vertical $\angle$s at R are equal in the two $\triangle$s ROy and RNQ, and the $\angle$ Q is common to both the $\triangle$s QON and RNQ: Consequently the three Triangles QON, QNR and ROy are similar; whence NQ $=$ NR, as also OR $=$ Oy; and, by the like Reason, Py $=$ PT.

By 4. 6 *Eucl. Elem.*

$$r\,(ON) \cdot\cdot\, x\,(NQ) :: x\,(NQ) \cdot\cdot\, \frac{xx}{r} = QR.$$

$$OQ - QR = r - \frac{xx}{r} = OR.$$

$$r\,(OQ) \cdot\cdot\, r - \frac{xx}{r}\,(OR) :: x\,(QT) \cdot\cdot\, x - \frac{x^3}{rr} = Ry.$$

$$NR + Ry + yP = x + x - \frac{x^3}{rr} + x = NP = b =$$

$3x - \frac{x^3}{rr}$; and, by multip'ying each Part by rr, you have

$brr = 3rrx - x^3$, or $x^3 - 3rrx = -brr$.

In

In this Equation the Cube of one Third the Co-efficient of x, added to the Square of half the known Quantity, gives the Sum $= -r^6 + \frac{b^2 r^4}{4}$, which is $\sqsupset$ 0, b being $\sqsupset 2r$; and therefore this Equation is of the same Nature with them, whose Roots, tho' included in *Cardan*'s Theorems, can't, notwithstanding, be easily extracted by, or out of them: But, since the Value of x can be found by the Scheme itself and Trigonometrical Calculations, that Case in Cubick Equations, wherein *Cardan*'s Method seems defective, is easily solvable by this:

Thus, if $x^3 - px = -q$. By comparing both Equations together, you will see that $p = 3rr$ and $q = brr$.

Now say, $\sqrt{\frac{1}{27}p^3} \cdot\cdot \frac{1}{2}q ::$ Radius .. S of an Angle.

That is, $r^3 \cdot\cdot \frac{rrb}{2} :: (r \cdot\cdot \frac{b}{2} ::$ O P .. P v ::) Radius ..

S $\angle$ P O v: And $\frac{1}{3}$ $\angle$ P O v = T O w; wherefore, since by 1 *Ax. Plane Trig.* Radius .. OT :: S $\angle$ TQ w .. T w, say, of Consequence, Radius .. $2\sqrt{\frac{p}{3}}$ (2 OT) :: S $\angle$ T O w ..

2Tw = TQ = QN = x sought.

Example.

If $x^3 - 12x = -10$; Quere x.

Log. $\frac{1}{2}q$ = L 5 = .6989700 - - - N. B. The Log. of Radius, *viz.* 10. is suppos'd to be added to this Number.

L $\sqrt{\frac{p^3}{27}}$ = L 8 = .9030900

9.7958800 = L - S 38°. 40′. 56″. = $\angle$ P O v, $\frac{1}{3}$ of which = 12°. 53′. 38″$\frac{2}{3}$ = $\angle$ T O w.

L - S $\angle$ T O w = 9.3485955

L $2\sqrt{\frac{1}{3}p}$ = L 4 = .6020600

9.9506555

L Radius = 10

$\bar{1}$.9506555 = Logarithm of .89260; that is to say, .89260 = x.

Again, 'tis plain that P N is not only the Chord of the Arc P Q N, but also of the Arc P H N; wherefore, x or T Q being = .89260, as before found, x or H I must be equal to some

some other Number, which, for to find, subtract the $\angle$ PON (or Arc PQN) from 360°, and $\frac{1}{3}$ the Remainder $= 60° - \frac{1}{3} \angle$ PON $= 60° - \angle$ TOw is $= \angle$ HOK; wherefore Radius ·· OH :: S $\angle$ HOK ·· HK; Consequently, Radius ·· 2 OH :: S $\angle$ HOK ·· 2 HK $=$ HI $= x$.

Operation.

$$\begin{array}{rrrl} ° & ' & '' & \\ 60 & 00 & 00 & \\ 12 & 53 & 38\frac{2}{3} & = \angle \text{TO}w. \\ \hline 47 & 06 & 21\frac{1}{3} & = \angle \text{HOK}. \end{array}$$

$$\text{L } 2\text{OH} = \text{L } 2\sqrt{\tfrac{1}{3}p} = \text{L } 4 = .6020600$$

$$\text{L-S } 47°.\ 06'.\ 21''\tfrac{1}{3}. = \underline{9.8648747}$$

$$10.4669347 = \text{Logarithm}$$

of $2.93045 = x$.

And PN may be the Chord of the * Arc PQNHPQN; Consequently add the $\angle$ TOw to 60°; and then it will be Radius ·· $2\sqrt{\frac{1}{3}p}$ :: S of the Sum of the $\angle$s TOw and 60° ·· x required.

Operation.

$$\begin{array}{rrr} ° & ' & '' \\ 60 & 00 & 00 \\ 12 & 53 & 38\frac{2}{3}. \\ \hline 72 & 53 & 38\frac{2}{3}. \end{array}$$

$$\text{L } 2\sqrt{\tfrac{1}{3}p} = .6020600$$

$$\text{L-S } 72°.\ 53'.\ 38''\tfrac{2}{3}. = \underline{9.9803500}$$

$$10.5824100 = \text{Logarithm of}$$

3.82305; that is to say 3.82305, with its proper Sign, is $= x$.

So that the three Values of x, in the Equation $x^3 - 12x = -10$, are .89260, 2.93045, and -3.82305: But if it were $x^3 - 12x = 10$; then x would be equal to 3.82305, -2.93045, and $-.89260$.

The learned Dr. *Halley* has, in Part of his Works, shewn a Method of resolving Numerically all other Sorts of Cubick Equations wanting the second Term by *Cardan*'s Theorems, and by the Tables of Sines, Tangents and Logarithms: But that being as easily, if not easier, done by *Cardan*'s Theorems, and the Logarithms only, I shall, at present, pass by that Method.

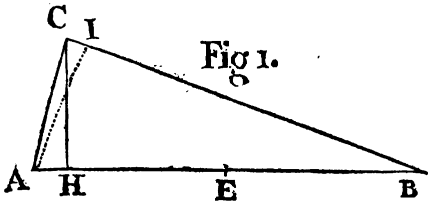
---

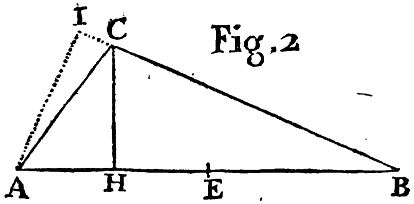
* *See Sir* Is. Newton's Arithm. Universalis, *Pag.* 236.

PROB.

PROB. X.

The Baſe of a Plane Triangle being given, as alſo the Perpendicular, and Angle oppoſite to the Baſe; to find the Triangle.





In the annex'd Triangles are given the Baſe AB $= 2b$ ($= 20$), the Perpendicular HC $= p$ ($= 7$); as alſo the Sine of the $\angle$ oppoſite to the Baſe, *viz.* of $\angle$ ACB $= m$ ($= 9.848077$, the ſaid $\angle$ACB being 80°, as in *Fig.* 1. or 100°, as in *Fig.* 2.) and conſequently the Co-ſine of $\angle$ ACB $= n$ ($= 1.736482 =$ Sine of 10°). 'Tis requir'd to find the other Sides and Angles.

From A let fall AI $\perp$ BC, produc'd if need be.

Biſect AB in E; and ſuppoſe HE $= x = ?$

Then AH $= b - x$.

And HB $= b + x$.

Suppoſe alſo BC $= y$.

Solution.

The Δs BCH and BAI are similar; wherefore

4. 6 *E. El.*	1	$y \,..\, 2b :: \begin{cases} b+x \,..\, \dfrac{2bb+2bx}{y} = BI. \\ p \,..\, \dfrac{2bp}{y} = AI. \end{cases}$
	2	$y \backsim \dfrac{2bb+2bx}{y} = CI = \dfrac{yy \backsim : 2bb+2bx :}{y}$
Ax. I. *Pl. Trig.*	3	$m \,..\, \dfrac{2bp}{y} :: n \,..\, \dfrac{2nbp}{my} = CI.$
$2 = 3.$	4	$\dfrac{yy \backsim : 2bb+2bx :}{y} = \dfrac{2nbp}{my}.$
$4 \times y.$	5	$yy \backsim : 2bb+2bx : = \dfrac{2nbp}{m}.$
That is	6	$yy = 2bb+2bx \pm \dfrac{2nbp}{m}.$
47. 1 *E. El.*	7	$yy = bb+2bx+xx+pp.$
6, 7.	8	$bb \pm \dfrac{2nbp}{m} = xx+pp.$
8 Reduc'd	9	$\surd : bb - pp \pm \dfrac{2nbp}{m} : = x.$

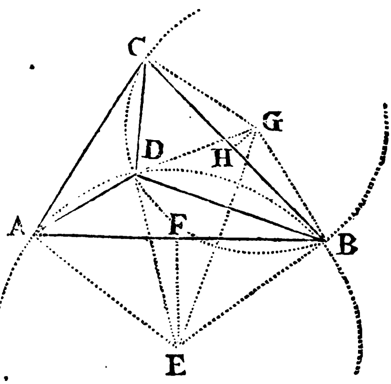
That is to say, if the given $\angle$ ACB be ⊐ L, as in *Fig.* 1. then $x = \surd : bb - pp + \dfrac{2nbp}{m} : (= \surd : 51 + 24.6857 + : = 8.6997 +)$; and, if the given $\angle$ ACB be ⊏ L, as in *Fig.* 2. then $x = \surd : bb - pp - \dfrac{2nbp}{m} : (= \surd : 51 - 24.6857 + : = 5.1297 +)$.

But if the given $\angle$ ACB had been = L, then n would be $= 0$; and consequently (*per* 9th Step) $x = \surd : bb - pp :$

x being thus found, you will have AH $= b - x$, and HB $= b + x$; and then the rest may be found by *Plane Trigonometry.*

PROB.

PROB. XI.



The Distances AB = 20000, AC = 15000, and BC 18000 between three Towers A, B and C (not standing in a straight Line) being given severally; as also a fourth Tower being suppos'd to stand within the Triangle ABC as at D; and the Measures of the Angles ADB = 127°, BDC = 105°, and CDA = 128°, being given severally; to find the Distances between the fourth Tower D and each of the other three; *viz.* the Measure of the Lines DA, DB and DC.

Construction.

On AB make an *Isosceles* △ AEB, whose ∠ AEB, or its Complement to 360° shall be = 2 ∠ ADB; on E as a Center, with the Radius AE or BE, describe the Circle ADB.

On BC make an *Isosceles* △ BGC, whose ∠ BGC, or its Complement to 360° shall be = 2 ∠ BDC: On G as a Center with the Radius BG or CG, describe the Circle BDC, cutting the former Circle in D (and B): Draw AD, BD and CD, and the Construction is perform'd.

Method of Calculation by Kersey.

On AB and BC let fall the Perpendiculars EF and GH and draw DE, DG and EG.

1. Subtract the given Angle ADB from two Right-angles (*viz.* from 180 Degrees) the Remainder shall be the Sum of the unknown Angles DAB and DBA, by 32. 1 *Eucl. Elem.*

2. Forasmuch as (by 20. 3 *Eucl. Elem.*) ∠ DEB = 2 ∠ DAB, and ∠ DEA = 2 ∠ DBA; it follows, that ∠ AEB = 2 ∠ DAB + 2 ∠ DBA; therefore in the △ FEB right-angled at F, the ∠ FEB (that is $\frac{1}{2}$ ∠ AEB) = ∠ DAB + ∠ DBA is given; and, by Supposition FB = $\frac{1}{2}$ AB is given; therefore the Semi-diameter EB = ED = EA shall be given also.

3. By arguing as before in 1st and 2d Steps ∠ HGB = ∠ HGC is given; also GD = GC = GB the Semi-diameter of the Circle GBDC is given.

 4. Because

4. Becauſe EF is $\perp$ AB, and $\angle$ FEB is given as before; therefore $\angle$ EBA, the Complement of $\angle$ FEB to a $\llcorner$ is given: Likewiſe $\angle$ GBC, the Complement of $\angle$ HGB to a $\llcorner$ is given; and the $\angle$ CBA is given; for it may be found out by the three given Sides AB, AC and BC; therefore $\angle$ GBE, the Sum of thoſe three Angles EBA, CBA and GBC is given.

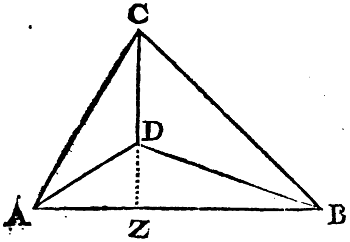
5. In the $\triangle$ GBE the Sides GB and EB (to wit, the Semi-diameters of the two Circles GBDC and EADB) are given ſeverally, as alſo the $\angle$ GBE comprehended by thoſe Sides; therefore the $\angle$ GEB is given alſo.

6. Becauſe the two $\triangle$s EGB and EGD have two Sides GB and EB equal to the two Sides GD, ED, *viz.* GB = GD, and EB = ED, alſo the Baſe GE common to both thoſe $\triangle$s; the $\angle$s contain'd under equal Right Lines ſhall be equal; *viz.* $\angle$ GEB = $\angle$ GED = $\frac{1}{2}$ $\angle$ DEB; but GED (= $\frac{1}{2}$ $\angle$ DEB) is given in the 5th Step, and (*per* 20. 3 *Eucl. El.*) $\angle$ DAB is = $\frac{1}{2}$ $\angle$ DEB (= $\angle$ GEB); therefore $\angle$ DAB is given.

Now in the $\triangle$ ADB there is given $\angle$ DAB, as alſo $\angle$ ADB, and the Side AB; therefore the Sides DB and DA (to wit, two of the Diſtances ſought) are given alſo.

7. And laſtly, In the $\triangle$ DAC there is given DA, as alſo AC, and $\angle$ ADC; therefore the third Diſtance ſought is given.

An Algebraical Solution of PROB. XI.



AB = b (= 20000), BC = c (= 18000), CA = d (= 15000), S $\angle$ ADB = f (= 7.986355 = S 127°.), Σ $\angle$ ADB = p (= 6.018150 = S 37°.), S $\angle$ CDB = g (= 9.659258 = S 105°.), S $\angle$ ADC = h (= 7.880108 = S 128°.)

S $\angle$ CAB = l (= 8.650611 + = S 59° 53′ 23″ −),
Σ $\angle$ CAB = k (= 5.016667 − = S 30 06 37 +),
S $\angle$ ABC = m (= 7.208842 − = S 46 07 39 +),
Σ $\angle$ ABC = n (= 6.930556 − = S 43 52 21 −),

being given, and the $\angle$ ADB being obtuſe, and the Angles CAB and ABC acute; 'tis required to find AD = a = ?

Draw DZ $\perp$ AB.

Solution.

Solution *.

Ax. II. *Pl. Trig.*	1	$b \cdot\cdot f :: a \cdot\cdot \frac{fa}{b} = S\angle DBA.$
	2	$\sqrt{:rr - \frac{ffaa}{bb}:} = \Sigma\angle DBA = S\angle BDZ = y.$
By *Theo.* 9	3	$\frac{ffa}{rb} + \frac{p}{r}y = S\angle ADZ.$
By *Theo.* 10	4	$\frac{f}{r}y - \frac{pf}{rb}a = S\angle DAZ.$
By *Theo.* 1	5	$\frac{lffa}{rrb} + \frac{lp}{rr}y - \frac{kf}{rr}y + \frac{kpfa}{rrb} = S\angle DAC.$
By *Theo.* 1	6	$\frac{m}{r}y - \frac{nfa}{rb} = S\angle DBC.$
Ax. II. *Plane Trig.*	7	$g \cdot\cdot c :: \frac{my}{r} - \frac{nfa}{rb} \cdot\cdot \frac{cm}{gr}y - \frac{cnfa}{grb} = CD.$
Ax. II. *Plane Trig.*	8	$d \cdot\cdot h :: \frac{cm}{gr}y - \frac{cnfa}{grb} \cdot\cdot \frac{hcm}{dgr}y - \frac{hcnfa}{dgrb} = S\angle CAD.$
8, 5.	9	$:\frac{hcm}{dg} - \frac{lp-kf}{r}: \times y = :\frac{hcnf}{dgb} + \frac{lff+kpf}{rb}: \times a.$

Now for $\frac{hcm}{dg} - \frac{lp-kf}{r}$ put t, and for $\frac{hcnf}{dgb} + \frac{lff+kpf}{rb}$ put w, and the 9th Equation will become

	10	$ty = wa = t\sqrt{:rr - \frac{ffaa}{bb}:}$
10 ⊕ 2.	11	$wwaa = tt \times :rr - \frac{ffaa}{bb}:$
11 Reduc'd	12	$a = \frac{trb}{\sqrt{:ttff+bbww:}}$

* Note, *The Theorems mention'd in this* Solution *are inserted in the following* Lemma.

(Viz.

(Viz. a is $=8283.368$ nearly, as may be computed from the last Step by the Help of the Logarithms; thus,

$$\begin{array}{rl|rl}
Lb = & .8965321 & & \\
Lc = & 4.2552725 & Ld = & 4.1760913 \\
Lm = & .8578655 & Lg = & .9849438 \\
\hline
Lbcm = & 6.0096701 & Ldg = & 5.1610351 \\
 & 5.1610351 & & \\
\hline
L:dg)bcm: = & .8486350 & \text{Number} & 7.057242
\end{array}$$

$$\begin{array}{rl}
Ll = & .9370468 \\
Lp = & .7794630 \\
\hline
Llp = & 1.7165098 \quad \text{Number } 52.06067 \\
Lk = & .7004152 \\
Lf = & .9023486 \\
\hline
Lkf = & 1.6027638 \quad \text{Number } 40.06488
\end{array}$$

$$lp - kf = 11.99579; \text{ wherefore}$$

$$r):lp-kf: = 1.199579$$

$$5.857663 = t.$$

$$\begin{array}{rl|rl}
Lbc = & 5.1518046 & & \\
Ln = & .8407681 & Ldg = & 5.1610351 \\
Lf = & .9023486 & Lb = & 4.3010300 \\
\hline
Lbcnf = & 6.8949213 & Ldgb = & 9.4620651 \\
 & 9.4620651 & & \\
\hline
L:dgb)bcnf: = & \bar{3}.4328562 & \text{Number} & .002709294
\end{array}$$

$$\begin{array}{rl}
Ll = & .9370468 \\
Lff = & 1.8046972 \\
\hline
Llff = & 2.7417440 \quad \text{Number } 551.7522. \\
Lkf = & 1.6027638 \\
Lp = & .7794630 \\
\hline
Lkfp = & 2.3822268 \quad \text{Number } 241.1164
\end{array}$$

$$lff + fkp = 792.8686; \text{ wheref.}$$

$rb)$

$rb) : lff + fkp : = .0039643 43$
$.002709294$

$.006673637 = w.$

$Lt = .7677243$
$Lr = 1.$
$Lb = 4.3010300$

$Ltrb = 6.0687543$

$Ltt = 1.5354486$
$Lff = 1.8046972$

$Lttff = 3.3401458$ Number 2188.496
$Lbb = 8.6020600$
$2Lw = Lww = \bar{5}.6487252$
$Lbbww = 4.2507852$ Number 17814.97

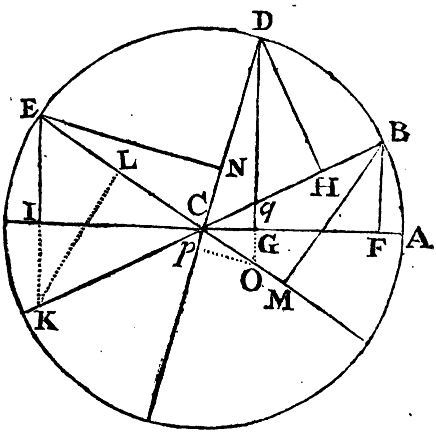
20003.466 =

$ttff + bbww.$

$Ltrb = 6.0687543$
$\frac{1}{2} L : ttff + bbww : = 2.1505526$

$La = 3.9182017$ Number 8283.368 = a nearly, as before, *Anſwer.*)

LEMMA.



Let AB and AD be two given Arcs of a Circle, the greater of which AD is leſs than a Quadrant; the Difference of the ſaid Arcs is DB. Let AE be another given Arc greater than a Quadrant; ſo as the Differences of the given Arcs AE and AB, *viz.* BE, and of the given Arcs AE and AD, to wit DE be, the former Difference greater, and the latter leſs than a Quadrant. Then, the Radii

dii and Sines of the said Arcs being drawn as in the annexed *Fig.* I say, as in the following *Theorems*.

Suppose $\begin{cases} b = \text{BF the Sine of the Arc AB; its Co-sine CF} = l. \\ c = \text{DG (which cuts CB in } q\text{) the Sine of the Arc AD;} \\ \quad \text{its Co-sine CG} = m. \\ a = \text{DH the Sine of the Arc BD (the Difference of the} \\ \quad \text{given Arcs AD and AB); its Co-sine CH} = x. \\ d = \text{EI the Sine of the Arc AE; its Co-sine IC} = n. \\ e = \text{BM the Sine of the Arc BE (the Difference of the} \\ \quad \text{given Arcs AE and AB); its Co-sine MC} = y. \\ i = \text{EN the Sine of the Arc DE (the Difference of the} \\ \quad \text{given Arcs AE and AD); its Co-sine CN} = z. \\ \text{And } r = \text{Radius.} \end{cases}$

Produce EI 'till it meets BC produc'd in K, and from K on CE let fall the ⊥ KL; produce also DG 'till it meets EC produc'd in O, and from O let fall Op ⊥ DC produc'd.

The △s CBF, CqG and DqH are similar; wherefore, by 4. 6 *Eucl. Elem.*

$$1.\ l \,..\, b :: m \,..\, \frac{bm}{l} = q\text{G}.$$

$$2.\ c - \frac{bm}{l} = \frac{cl - bm}{l} = \text{D}q.$$

$$3.\ r \,..\, l :: \frac{cl - bm}{l} \,..\, \frac{cl - bm}{r} = \text{DH} = a. \quad \textit{Theorem } 1.$$

$$4.\ l \,..\, r :: m \,..\, \frac{rm}{l} = \text{C}q.$$

$$5.\ r \,..\, b :: \frac{cl - bm}{l} \,..\, \frac{bcl - bbm}{rl} = q\text{H}.$$

$$6.\ \frac{rm}{l} + \frac{bcl - bbm}{rl} = \frac{:rr - bb: \times m + bcl}{rl} =$$

$$\frac{llm + bcl}{rl} \text{ (because } rr - bb \text{ is equal to } ll) = \frac{lm + bc}{r}$$

$$= \text{CH} = x. \quad \textit{Theor. } 2.$$

The Theorems in the 3d and 6th Steps are the same with the Reverend Mr. *Pierre Gouli*'s 2d and 3d *Lemmas* respectively; and may be worded thus:

Two Arcs being given, the greater of which is leſs than a Quadrant; if from the Rectangle, or Product of the Sine of the Greater and Co-ſine of the Leſs, the Rectangle of the Sine of the Leſſer, and Co-ſine of the Greater Arc be ſubtracted, and the Remainder divided by Radius, the Quotient will be the Sine of the Difference of the two Arcs. And, if the Sum of the Rectangles of their Co-ſines and Sines be divided by Radius, the Quotient will give the Co-ſine of the Difference of the two Arcs.

7. But $\frac{cl-bm}{r} \times \frac{l}{r} + \frac{lm+bc}{r} \times \frac{b}{r} \left(= \frac{ll+bbc}{rr} c\right) = c$; that is to ſay $\frac{al+xb}{r} = c =$ DG. *Theor.* 3.

8. And $\frac{lm+bc}{r} \times \frac{l}{r} - \frac{cl-bm}{r} \times \frac{b}{r} \left(= \frac{ll+bb}{rr} m\right) = m$; viz. $\frac{xl-ab}{r} = m =$ CG. *Theor.* 4.

Again, the Δs CBF and CKI, as alſo the Δs CEI and KEL, and the Δs CBM and CKL are ſimilar; therefore, by 4 6 *Eucl. Elem.*

9. $l \cdot\cdot n :: \begin{cases} b \cdot\cdot \frac{bn}{l} = \text{IK} \\ r \cdot\cdot \frac{rn}{l} = \text{KC} \end{cases}$

10. $d + \frac{bn}{l} = \frac{dl+bn}{l} =$ EK.

11. $r \cdot\cdot \frac{dl+bn}{l} :: \begin{cases} n \cdot\cdot \frac{dln+bnn}{rl} = \text{KL}. \\ d \cdot\cdot \frac{ddl+bdn}{rl} = \text{LE}. \end{cases}$

12. $r - \frac{ddl+bdn}{rl} = \text{LC} = \frac{nnl-bdn}{rl}$

13. $\frac{rn}{l} \cdot\cdot r :: \begin{cases} \frac{dln+bnn}{rl} \cdot\cdot \frac{dl+bn}{r} = e = \text{BM}. \textit{ Theo. } 5. \\ \frac{nnl-bdn}{rl} \cdot\cdot \frac{nl-bd}{r} = y = \text{MC}. \textit{ Theor. } 6. \end{cases}$

14. But $\frac{dl+bn}{r} \times \frac{l}{r} - \frac{nl-bd}{r} \times \frac{b}{r} \left(= \frac{ll+bb}{rr} d\right) = d$;

*E e that

that is to say $\frac{el - yb}{r} = d = EI$. *Theor.* 7.

15. And $\frac{dl + bn}{r} \times \frac{b}{r} + \frac{nl - bd}{r} \times \frac{l}{r} \left(= \frac{bb + ll}{rr} n \right) = n$;

viz. $\frac{eb + yl}{r} = n = IC$. *Theor.* 8.

Again, the Δs CEI and COG, as alſo the Δs CDG and ODp, and the Δs COp and CEN are ſimilar; therefore, by 4. 6 *Eucl. Elem.*

$$16.\ n \cdot\cdot m :: \left\{ \begin{array}{l} r \cdot\cdot \frac{rm}{n} = CO. \\ d \cdot\cdot \frac{dm}{n} = GO. \end{array} \right.$$

$$17.\ c + \frac{dm}{n} = \frac{cn + dm}{n} = DO.$$

$$18.\ r \cdot\cdot \frac{cn + dm}{n} :: \left\{ \begin{array}{l} m \cdot\cdot \frac{cmn + dmm}{rn} = Op. \\ c \cdot\cdot \frac{ccn + cdm}{rn} = Dp. \end{array} \right.$$

$$19.\ \frac{ccn + cdm}{rn} - r = \frac{cdm - : rr - cc : \times n}{rn} = \frac{cdm - mmn}{rn} = Cp.$$

$$20.\ \frac{mr}{n} \cdot\cdot r :: \left\{ \begin{array}{l} \frac{cmn + dmm}{rn} \cdot\cdot \frac{cn + dm}{r} = i = (EN.\ \mathit{Th.}\ 9. \\ \frac{cdm - mmn}{rn} \cdot\cdot \frac{cd - mn}{r} = z = (CN.\ \mathit{Th.}\ 10. \end{array} \right.$$

The Theorems in the 20th Step are the ſame with the Reverend Mr. *Gould*'s 4th and 1ſt *Lemma's*, and may be thus expreſſed in Words:

Two Arcs being given, the greater of which is more than a Quadrant; and the other, as alſo the Difference of the two given Arcs, are each of them leſs than a Quadrant: If the Sum of the Rectangles of the Sine of the one into the Coſine of the other Arc be divided by Radius, the Quotient will be the Sine of the Difference of the two Arcs. And, if from the Rectangle of their Sines the Rectangle of their Coſines

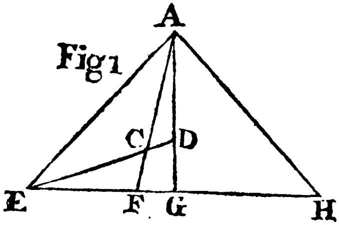
ſines be ſubtracted, and the Remainder divided by Radius, the Quotient will be the Co-ſine of the Difference of the two Arcs.

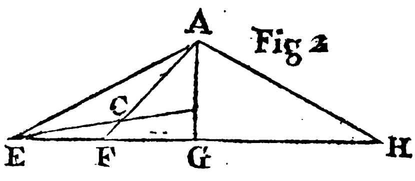
21. But $\frac{cn + dm}{r} \times \frac{m}{r} + \frac{cd - mn}{r} \times \frac{c}{r}$ $\left(= \frac{mm + cc}{rr} d\right) = d$; that is to ſay $\frac{im + zc}{r} = d =$ EI. *Theor.* 11.

And $\frac{cn + dm}{r} \times \frac{c}{r} - \frac{cd - mn}{r} \times \frac{m}{r} \left(= \frac{cc + mm}{rr} n\right) = n$; that is $\frac{ic - zm}{r} = n = $ IC. *Theorem.* 12.

PROB. XII.

Which is the Reverend Mr. P. Gould's *Problem.*





AH = AE, GH = GE and EC = AC.

AG = c is given; AD = $\frac{2c}{3}$ ∵ GD = $\frac{c}{3}$

Sine of ∠ FAH = s, and its Co-ſine = p are given:

It is required to find FG by an Equation not exceeding a Biquadratick.

This Problem admits of three Caſes; for the given ∠ FAH is either leſs or greater than a Right-angle, or elſe it is equal to a Right-angle.

Solution of the First and Second Cases.

Note, As to the Signs + and — one over the other in this Solution, that the upper Sign relates and belongs to the 1st Case and 1st Fig. and the Lower to the 2d Case and 2d Fig.

Note also, That the Theorems mention'd in this Solution are inserted in the preceding *Lemma*.

Suppose	1	$AF = x$, and $FG = y = ?$
Ax. I. *Pl. Trig.*	2	$x \cdot\cdot r :: \begin{cases} y \cdot\cdot \dfrac{ry}{x} = S \angle FAG. \\ c \cdot\cdot \dfrac{cr}{x} = S \angle AFG. \end{cases}$
By *The.* $\begin{cases} 1 \\ 9 \end{cases}$	3	$\dfrac{scr}{rx} \mp \dfrac{pry}{rx} = \dfrac{sc \mp py}{x} = S \angle GAH = S \angle GAE.$
Theo. $\begin{cases} 2 \\ 10 \end{cases}$	4	$\dfrac{sry \pm pcr}{rx} = \dfrac{sy \pm pc}{x} = S \angle GHA = S \angle GEA.$
Theor. 1.	5	$\dfrac{sc \mp py}{x} \times \dfrac{cr}{xr} - \dfrac{ry}{x} \times \dfrac{sy \pm pc}{rx} = \dfrac{scc \mp 2cpy - syy}{xx} = S \angle CAE = S \angle CEA.$
Theor. 2.	6	$\dfrac{sy \pm pc}{x} \times \dfrac{cr}{xr} + \dfrac{sc \mp py}{x} \times \dfrac{ry}{xr} = \dfrac{2csy \pm ccp \mp pyy}{xx} =$ Co-sine $\angle CAE$.
Theor. 1.	7	$\dfrac{sy \pm pc}{x} \times \dfrac{\mp pyy + 2csy \pm ccp}{xxr} - \dfrac{sc \mp py}{x} \times \dfrac{scc \mp 2cpy - syy}{xxr} = S \angle DEG = \dfrac{\mp 2spy^3 + 3cssy^2 - 3cppy^2 \pm 6ccspy + p^2c^3 - s^2c^3}{rx^3}.$

Ax. II.

Ax.II.Pl. Trig.	8	$\frac{scc \mp 2cpy - syy}{xx} \therefore \frac{2c}{3} :: \frac{sc \mp py}{x}$ $\therefore \frac{2ccsx \mp 2cpyx}{3ccs \mp 6cpy - 3syy} = \text{DE}.$
Ax.I. Pl. Trig.	9	$\frac{2ccsx \mp 2cpyx}{3ccs \mp 6cpy - 3syy} \therefore r :: \frac{c}{3} \therefore$ $\frac{rccs \mp 2rcpy - rsyy}{2csx \mp 2pyx} = \text{S} \angle \text{DEG}.$
7, 9.	10	$\mp 2spy^3 + 3cssyy - 3cppyy \pm 6ccspy + c^3pp - c^3ss = \frac{rrccsxx \mp 2rrcpyxx - rrsyyxx}{2cs \mp 2py}$
47. 1 *Euc. Elem.*	11	$cc + yy = xx.$
10, 11.	12	$\mp 2spy^3 + 3cssyy - 3cppyy \pm 6ccspy + c^3pp - c^3ss = \frac{r^2c^4s \mp 2rrc^3py * \mp 2rrcpy^3 - rrsy^4}{2cs \mp 2py}$
12 Reduc'd.	13	$\left.\begin{array}{l} y^4 \text{ - - - - -} \\ + y^3 \times \frac{\mp 10cssp \pm 6cp^3 \pm 2rrcp}{4spp + rrs} \\ + y^2 \times \frac{6ccss - 18c^2pp}{4pp + rr} \\ + y \times \frac{\pm 2rrc^3p \pm 14c^3ssp \mp 2c^3p^3}{4spp + rrs} \end{array}\right\} = \frac{rrc^4 + 2c^4ss - 2c^4pp}{4pp + rr}.$

Case 3.

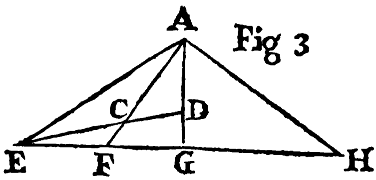


Fig 3

Case 3. When $\angle$ FAH is $= \llcorner$.

Then S $\angle$ FAH $= s$ will become $= r$, and its Co-sine $= p$ will become $= 0$: And then an Equation expressing the Value of FG $= y$, deducible from the Thirteenth Step, is $y^4 + yy \times \frac{6ccrr}{rr} = \frac{rrc^4 + 2c^4rr}{rr}$; which Equation, when reduc'd, gives $y = c\sqrt{:\sqrt{12} - 3:}$

A Numerical Example of Case 3.

Let AG ($= c$) be $= 3$, GD $\left(= \frac{c}{3}\right) = 1$, and $\angle$ FAH $= \llcorner$, &c.

Then $y = 3\sqrt{:\sqrt{12} - 3:} = 3\sqrt{:3.4641016151,\ \&c. - 3:} = 2.043750 + =$ FG.

Proof.

AH = AE, GH = GE. [*See the last* Fig.]

AG = 3, GD = 1, and ∵ AD = 2, FG = 2.043750 +, and $\angle$ FAH $= 90°$.

Required to shew that $\angle$ EAC $=$ $\angle$ AEC; *viz.* that EC is = AC.

1. In Δ FAG. As GA 3 - - - - - - 0.4771212

Is to Radius - - - - - 10.

So is FG 2.0437 - - - 0.3104171

To T of $\angle$A $34° \; 16'$ − - - - - - - - 9.8332959

90 00 = $\angle$ FAH.

34 16− = $\angle$ FAG.

55 44+ = $\angle$ GAH = $\angle$ AFG = $\angle$GAE.

34 16− = $\angle$ FAG = $\angle$ GHA = $\angle$ GEA

21 28+ = $\angle$ EAF.

2. In

2. In the $\triangle$ EAG. As S $\angle$ E 34°. 16′. - - 9.7505434

In the $\triangle$ EAG. As S $\angle$ E 34°. 16′. - -	9.7505434
Is to AG 3 - - - - - - - - - -	0.4771212
So is S $\angle$ A 55°. 44′. - - - -	9.9172040
	10.3943252
To EG 4.4030 + - - - - - -	0.6437818

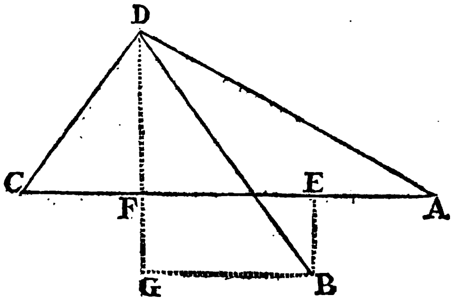
3. In the $\triangle$ DEG. As EG 4.4030 + - - - 0.6437818

In the $\triangle$ DEG. As EG 4.4030 + - - -	0.6437818
Is to Radius - - - - - - - - -	10.
So is DG 1 - - - - - - - -	0.
To T of $\angle$ E 12°. 47′. + - -	9.3562182

34° 16′ − = $\angle$ AEG.
12 47 + = $\angle$ DEG.

21 28 + = $\angle$ AED, or $\angle$ AEC, which is = $\angle$ EAF, or $\angle$ EAC above found, as was required to be ſhewn.

PROB. XIII.



Three Points A, B and C being given; to find another Point D in the ſame Plane, from which three ſtraight Lines being drawn to the given Points, thoſe ſtraight Lines may be to one another in any given Ratio's: Suppoſe DA to DB to DC, as m (3. 4) to n (3) to p (2).

Suppoſe it done, and D, in the annex'd *Fig.* the Point required.

The Lines DA, DB and DC being drawn, join any two of the three given Points, ſuppoſe A and C, by drawing the ſtraight Line AC; on which, from the third given Point B, and from the ſought one D, let fall the $\perp$s BE and DF.

Now let the Given Lines AC be = b (= 21)
AE = c (= 6)
EB = d (= 4)

And

And suppose the unknown Lines $AF = x$
and $FD = y$
Then will FC be $= b - x$
And $FE = x - c$

47. 1 E. Elem.

1	$ADq = xx + yy$.
2	$CDq\ (= FCq + FDq) = bb - 2bx + xx + yy$.
3	$BDq\ (= {}^{*}\overline{DF+EB}\vert^2 + EFq) = yy + 2dy + dd + xx - 2cx + cc$.

By *Hypothesis*, $m \cdot\cdot DA :: n \cdot\cdot DB :: p \cdot\cdot DC$; Consequently,

	4	$mm \cdot\cdot xx + yy :: nn \cdot\cdot \frac{nn}{mm} \times : xx + yy : = DBq = yy + 2dy + dd + xx - 2cx + cc$.
	5	$mm \cdot\cdot xx + yy :: pp \cdot\cdot \frac{pp}{mm} \times : xx + yy : = DCq = bb - 2bx + xx + yy$.
5,	6	$xx + yy = mm \times \frac{2bx - bb}{mm - pp}$.
4,	7	$xx + yy = mm \times \frac{2cx - 2dy - cc - dd}{mm - nn}$

Now if, in order to shorten the Operation, you write for $mm - pp\ (= 7.56)\ f$, and for $mm - nn\ (= 2.56)\ g$, you'll have from the 6th and 7th Steps.

8	$\frac{2gbx - gbb}{f} = 2cx - cc - dd - 2dy$.

Again, for Brevity's sake, for $-\frac{gb}{f} + c\ (= -\frac{20}{9})$ put $-h$; and for $\frac{gbb}{f} - cc - dd$ put $2dk$ ($= 97\frac{2}{9}$; consequently, $k = 12\frac{1}{8}$) and the 8th Step will become

* *For producing* DF *to* G, *making* $FG = EB$; $DF + FG$ *will be* $= DG$, *and* $EF = BG$.

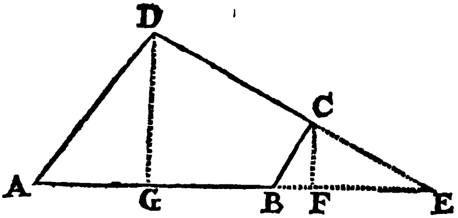
	9	$-2bx+2dk=2dy.$
$9\div 2d$	10	$-\frac{b}{d}x+k=y.$
10 ☉ 2.	11	$\frac{bb}{dd}xx-\frac{2bk}{d}x+kk=yy.$
6, 11.	12	$xx+\frac{bb}{dd}xx-\frac{2bk}{d}x+kk=mm\times\frac{2bx-bb}{f}.$
12.	13	$:f+\frac{fbb}{dd}:\times xx-:\frac{2fbk}{d}+2mmb:\times x=-fkk-mmbb.$

Laſtly, For the Coefficient of xx in the 13th Step put l $\left(=\frac{2443}{300}\right)$, for that of x put $-2lt$ $\left(=-536.62;\right.$ conſequently $\left.t=\frac{80493}{2443}\right)$, and for the abſolute Quantity therein put $-lw$ $\left(=-6217.05;\right.$ conſequently $\left.w=\frac{1865115}{2443}\right)$; then that Step will become

	14	$lxx-2ltx=-lw.$
14 Re-duc'd.	15	$x=t\pm\sqrt{:tt-w:}\left(=\frac{80493}{2443}\pm\frac{1}{2443}\sqrt{:6479123049-4556475945:}=50\frac{7191}{2443}\right.$ or $\left.15.\right)$

x being thus found, y will be had by the 10th Step $=-\frac{b}{d}x+k$ $(=-1\frac{2034}{7094}$, or $8)$: And by the Quantities tities x and y, viz. AF and FD thus known, the Point D ſought will be determin'd.

PROB. XIV.



Given the Angles, Perimeter and Area severally of a Trapezium; it is required to determine its Sides.

Suppose it done, and represented by the annex'd Trapezium ABCD.

Let $S \angle A = c$ ($= S\ 53^\circ.\ 08'. - = 8$), $S \angle B = d$ $\left(= S\ 118^\circ.\ 04'. + = \frac{150}{17}\right)$, $S \angle C = f$ ($= S\ 90^\circ.\ 00'. = 10$), and $S \angle D = g$ $\left(= S\ 98^\circ.\ 48'. - = \frac{168}{17}\right)$; also the Perimeter $AB + BC + CD + DA = b$ ($= 360$); and Area of the Trapezium $ABCD = a$ ($= 6900$).

Produce any two of the Trapezium's Sides which are opposite to one another, as DC and AB, till they meet in the Point E.

Then in the Δs ADE and BCE you have all the Angles given, and consequently the $\angle E$ ($= 28^\circ.\ 04'. +$) for whose Sine put h $\left(= \frac{80}{17}\right)$.

From the Points C and D on EA (produc'd if need be) let fall the $\perp$s CF and DG.

Now suppose $AD = x$ and $BC = y$.

Ax. II. *Pl. Trig.*	1	$h \cdot\cdot x :: \begin{cases} g \cdot\cdot \dfrac{gx}{h} = AE. \\ c \cdot\cdot \dfrac{cx}{h} = DE. \end{cases}$
	2	$h \cdot\cdot y :: \begin{cases} f \cdot\cdot \dfrac{fy}{h} = BE. \\ d \cdot\cdot \dfrac{dy}{h} = CE. \end{cases}$
$\because$	3	$\dfrac{gx - fy}{h} = AB.$
And	4	$\dfrac{cx - dy}{h} = CD.$
3, 4, and *Hypoth.*	5	$x + y + \dfrac{gx - fy + cx - dy}{h} = b.$

Now,

Now, for $1+\frac{g+c}{h}$ put $k\ (=4.8)$; also for $1-\frac{f+d}{h}$ put $-kl\left(=-3, \text{ then } l=\frac{5}{8}\right)$; and for b put km, (then $m=75$); then you'll from the 5th Step have

	6	$kx-kly=km.$
$\because$	7	$x=ly+m$; consequently $xx=llyy+2lmy+mm.$
Ax. I. Pl. Trig.	8	$r\cdot\cdot x::c\cdot\cdot\frac{cx}{r}=DG.$
	9	$r\cdot\cdot y::d\cdot\cdot\frac{dy}{r}=CF.$
$\because$	10	$\frac{gx}{h}\times\frac{cx}{2r}=\frac{cgxx}{2hr}=$ Area of $\Delta ADE.$
And	11	$\frac{fy}{h}\times\frac{dy}{2r}=\frac{dfyy}{2hr}=$ Area of $\Delta BCE.$
10, 11.	12	$\frac{cgxx-dfyy}{2hr}=a.$
7, 12.	13	$\frac{cgllyy+2cglmy+cgmm-dfyy}{2hr}=a.$

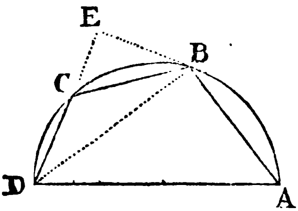
For $\frac{cgll-df}{2hr}$ put $-n\left(=-\frac{39}{64}\right)$, for $\frac{2cglm}{2hr}$ put $2np\left(=78\tfrac{3}{4}, \text{ then } p=\frac{840}{13}\right)$, and for $a-\frac{cgmm}{2hr}$ put $nq\left(=2175; \text{ then } q=\frac{46400}{13}\right)$; then from the 13th Step you'll have

	14	$-nyy+2npy=nq.$
14 Reduc'd	15	$y=p\pm\sqrt{:pp-q:}\left(=\frac{840\pm\sqrt{:705600-603200:}}{13}=89\tfrac{3}{13} \text{ or } =40.\right)$

 Hence

Hence two Trapezia may be found, which anfwer the Conditions required by the *Problem.* (viz. y being $= 89\frac{3}{13}$, x or AD $= ly + m$, *per* 7th Step, will be $= 130\frac{10}{13}$; then AB $= \frac{gx - fy}{h}$, by the third Step, $= 85$, and CD $= \frac{cx - dy}{h}$, *per* 4th Step, $= 55$: And y being $= 40$, then AD will be $= 100$, AB $= 125$, and CD $= 95$.)

PROB. XV.



The three Chords, or Subtenfes, of three Arches compleating a Semi circle being feverally given; thence to find the Diameter of that Circle.

Given AB $= b$ (78), BC $= c$ (66), and CD $= d$ (50);

Required to find AD $= a =$?

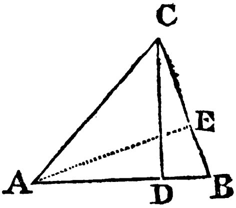
Draw the Diagonal BD, then from B let fall BE $\perp$ DC produc'd.

Now the $\angle$ BAD $+ \angle$ BCD $= 2 \llcorner$, by 22. 3 *Eucl. Elem.* confequently $\angle$ BAD $= \angle$ BCE. Again, $\angle$ ABD $= \llcorner$, by 31. 3 *Eucl. Elem.* and $\because$ it is $= \angle$ BEC: Confequently the $\triangle$s ABD and CEB are fimilar; wherefore

4. 6 *E. El.*	1	$a\,(AD) \cdot\cdot\, b\,(AB) :: c\,(BC) \cdot\cdot\, \frac{bc}{a} = CE.$
$\because$	2	$d + \frac{bc}{a} = DE.$
47. 1 *Euc. Elem.*	3	$aa - bb = BDq.$
	4	$BDq - DEq = BEq = aa - bb - dd - \frac{2dbc}{a} - \frac{bbcc}{aa}.$
	5	$cc - \frac{bbcc}{aa} = BEq.$
4, 5.	6	$aa - bb - dd - \frac{2dbc}{a} = cc.$
6 Reduc'd	7	$a^3 \begin{matrix} - bb \\ - cc \\ - dd \end{matrix}\, a = 2bcd$ (viz. $a^3 - 12940\,a =$ 514800, an adfected Cubick Equation, wherein the Value of a is $130 =$ AD.)

PROB.

PROB. XVI.



In an Acute-angled Plane Triangle the Sum of the Sides, as alſo the Perpendicular let fall upon the Baſe, likewiſe the Sine of the Angle oppoſite to the Baſe, and conſequently its Co-ſine being ſeverally given; thence to find the Triangle.

Suppoſe ABC to be the Triangle ſought, AB its Baſe, and CD the ⊥ let fall on the Baſe; then on CB let fall the ⊥ AE.

$$\left.\begin{array}{l} \text{Now } AC + CB = b\ (= 178) \\ CD = c\left(= 76\tfrac{4}{41} = \dfrac{3120}{41}\right) \\ S\angle ACB = m\ (= 8 = S \text{ of } 53^\circ.\ 07' + \text{ the} \\ \text{Radius } r \text{ being} = 10) \\ \Sigma\angle ACB = S\angle CAE = n\ (= 6 = S\ 36^\circ.\ 52' +) \end{array}\right\}\text{Given.}$$

$AC = a$, and conſequently $BC = b - a$, and $AB = y$ $= ?$ Required.

Solution.

Ax. I. *Pl. Trig.*	1	$r \,..\, a :: \begin{cases} m \,..\, \dfrac{ma}{r} = AE. \\ n \,..\, \dfrac{na}{r} = EC. \end{cases}$
∴	2	$b - a - \dfrac{na}{r} = EB = \dfrac{br - ra - na}{r}.$
4.6 *E. El.*	3	$c \,..\, b - a :: \dfrac{ma}{r} \,..\, \dfrac{m}{cr} \times : ba - aa := y$ $= AB.$
3,	4	$\dfrac{cry}{m} = ba - aa.$

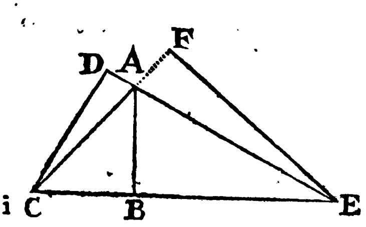
By 47. 1 *Eucl. Elem.* $AEq + EBq = ABq$; that is,

	5	$\frac{mmaa + bbrr - 2brra - 2brna + rraa + 2nraa + nnaa}{rr} = yy.$
But	6	$mm + nn = rr.$
5, 6.	7	$\frac{bbrr - 2brra - 2bnra + 2rraa + 2nraa}{rr} = yy.$
7,	8	$2bra + 2bna - 2raa - 2naa = bbr - ryy.$
8 ÷ : 2r + 2n :	9	$ba - aa = \frac{bbr - ryy}{2r + 2n}.$
4 = 9.	10	$\frac{cry}{m} = \frac{bbr - ryy}{2r + 2n}.$
10,	11	$bbm - myy = 2cry + 2cny = 2cpy$ (putting $r + n = p$.)
11 Reduc'd	12	$y = -\frac{cp}{m} + \sqrt{:bb + \frac{ccpp}{mm}:}$ ($= 82$).

The Value of y being thus known, the Values of a or AC ($=$ 100 or 78) may be found by the latter Part of the 3d Step; And then $CB = b - a$ ($=$ 78 or 100).

The Methods of solving the two other Cases, *viz.* wherein the given ∠ ACB is either right or obtuse, are so obvious from the above *Solution*, that I need not farther insist upon them.

PROB. XVII.



In a Plane Triangle the three Perpendiculars being severally given; 'tis required to find its Sides.

It not yet appearing from the *Data* whether the Triangle be a ∟d, an acute or an obtuse-angled one: But it being certain, that the shortest Perpendicular falls within any, or either Sort of △s: Let therefore the shortest given Perpendicular be AB (in the annex'd *Fig.*) $= b$ (8), the shortest but

but one given Perpendicular CD $= c$ $(9\frac{15}{17})$, and the longeſt or greateſt given Perpendicular EF $= d$ (16.8): And ſuppoſe CE $= a = ?$

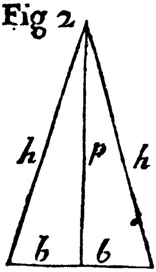
Then	1	$\frac{1}{2}ba =$ Area of the $\triangle$ ACE.
	2	$\frac{AE \times c}{2} =$ Area of the $\triangle$ ACE $= \frac{ba}{2}$.
1, 2.	3	$AE = \frac{b}{c}a$.
In like Manner	4	$AC = \frac{b}{d}a$.
47. 1 *Euc. Elem.*	5	$\frac{bb}{dd}aa - bb = CBq$.
Ax. IV. *Plane Trig.*	6	$a \cdot\cdot \frac{b}{c}a + \frac{b}{d}a :: \frac{b}{c}a - \frac{b}{d}a \cdot\cdot \frac{bb}{cc}a - \frac{bb}{dd}a = BE - CB$.
	7	$\frac{a}{2} - \frac{bba}{2cc} + \frac{bb}{2dd}a = CB = fa$, putting f equal to $\frac{1}{2} - \frac{bb}{2cc} + \frac{bb}{2dd}$, $\left(= \frac{2}{7}a.\right)$
7 ⊙ 2 = 5.	8	$ffaa = \frac{bb}{dd}aa - bb$.
8 Reduc'd	9	$\frac{bd}{\sqrt{:bb - ffdd:}} = a$ (= 21.)

The Value of $a =$ CE being thus found, the Values of AE and AC (the two other Sides required of the $\triangle$) will be given by the 3d and 4th Steps. (Viz. AE = 17, and AC = 10.)

PROB.

PROB. XVIII.

Fig 2



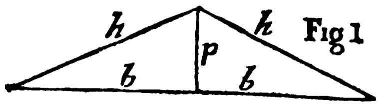


Fig 1

'Tis required to find two ſuch *Iſoſceles* Triangles, that the Area of each Triangle being $= s$, ſhall be to its Perimeter in a given Ratio, *viz.* As *n* to *m*.

1. Suppoſing the Poſſibility of finding two ſuch Δs as are required by the *Problem*, Imagine them to be repreſented by the annex'd *Figures*, in each of which I will ſuppoſe the Perpendicular $= p = ?$

2. Then b will be $= \frac{s}{p}$ $\left(\text{for } \frac{p \times 2b}{2} = pb = s \because b = \frac{s}{p}\right)$

3. *Per* 47. 1 *E. El.* $h = \sqrt{:pp + \frac{ss}{pp}:} = \frac{\sqrt{:p^4 + ss:}}{p}$.

4. $\because$, *Per* Prob. $s \cdot\cdot \frac{2\sqrt{:p^4 + ss:}}{p} + \frac{2s}{p} (= 2h + 2b) :: n \cdot\cdot m$.

5. Conſequently $sm = \frac{2n\sqrt{:p^4 + ss:}}{p} + \frac{2ns}{p} = \frac{msp}{p}$.

6. Which Equation, when reduc'd, gives $p^3 - \frac{mmss}{4nn} p + \frac{mss}{n} = 0$.

Now when the Cube of one third Part of $\frac{mmss}{4nn}$ is not leſs than the Square of one half of $\frac{mss}{n}$, this Equation will have two affirmative Roots, and conſequently the *Problem* may be ſolv'd: As for

Example.

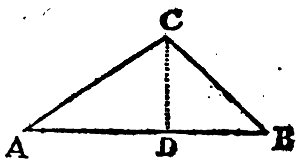
Example.

If s be $= 420$, and $m = n$, then the above Equation will become $p^3 - 44100p + 176400 = 0$, in which the two Roots or Values of p are $4 +$ (*viz.* $4.00145289 +$) and $208 -$: Supposing therefore p in *Fig.* 1. to be $= 4 +$; then b $\left(= \frac{s}{p}\right)$ will be $= \frac{420}{4+} = 105 -$; also $h (= \sqrt{: pp + bb :})$ $= 105 +$: Consequently $2b + 2h =$ the Perimeter $= 210 - + 210 + = 420$: Whence 420 (= the Area) $\cdot\cdot$ 420 (= the Perimeter) :: $n = m \cdot\cdot m = n$.

Again, the other Value of p being $208 -$ (viz. p in *Fig.* 2. $= 208 -$) then $b \left(= \frac{s}{p}\right)$ will be $= \frac{420}{208-} = 2 +$; also h $(= \sqrt{: pp + bb :}) = 208 -$; consequently $2b + 2h =$ Perimeter $= 4 + + 416 - = 420$: Wherefore the Area and Perimeter of *Fig.* 2. too are to each other in a Ratio of Equality, as was required by this Example.

Lemma.

If from half the Sum of the three Sides of any Plane Triangle you subtract each of the three Sides severally, and then multiply those three Remainders and the said half Sum together; I say, the Square-Root of the Product shall be the Area of the Triangle.



In the annex'd Triangle let $AB = b$.
$BC = c$.
$CA = d$.
And the Area of the Triangle $ABC = a$.

I say, the Square-Root of $\frac{-b+c+d}{2} \times \frac{b-c+d}{2} \times \frac{b+c-d}{2} \times \frac{b+c+d}{2} = \frac{1}{4}\sqrt{: 2bbcc + 2bbdd + 2ccdd - b^4 - c^4 - d^4 :}$ is $= a$.

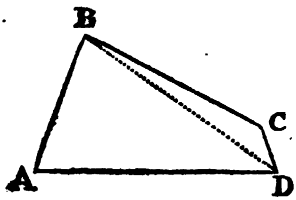
Demonstration.

On the longest Side AB of the $\triangle$ let fall the $\perp$ CD; then

4. *Ax. Pl.Trig.*	1	$b \,..\, d+c :: d-c \,..\, \frac{dd-cc}{b} = AD-DB.$
	2	$\frac{b}{2}+\frac{dd-cc}{2b}=\frac{bb+dd-cc}{2b}=AD.$
47. 1 *Euc. Elem.*	3	$\sqrt{:dd-\frac{2bbdd-2bbcc-2ccdd+b^4+c^4+d^4:}{4bb}}=DC.$ $=\frac{\sqrt{:2bbdd+2bbcc+2ccdd-b^4-c^4-d^4:}}{2b}$
$3\times\frac{b}{2}$	4	$\frac{1}{4}\sqrt{:2bbdd+2bbcc+2ccdd-b^4-c^4-d^4:}=a.$

W. W. D.

PROB. XIX.



To make of the four given Lines $AB=b$ (30), $BC=c$ (41), $CD=d$ (9), and $DA=f$ (50) the Trapezium ABCD, whose Area may be equal to a given Quantity $=g$ (780); but then g must be such as the Thing required may be possible to be done.

Supposing it done, and truly delineated by the annex'd *Fig.* Draw the Diagonal BD, which suppose $=x=?$; then, by the precedent *Lemma*,

1. The Area of the $\triangle$ ABD is $=\frac{1}{4}\sqrt{:2bbff\,{+2bb \atop +2ff}\,xx\,{-b^4 \atop -f^4}-x^4:}$

2. And the Area of the $\triangle$ BCD is $=\frac{1}{4}\sqrt{:2ccdd\,{+2cc \atop +2dd}\,xx\,{-c^4 \atop -d^4}-x^4:}$

Now for $2bb+2ff$ ($=6800$) put h, and for $b^4+f^4-2bbff$ ($=\overline{bb\backsim ff}|^2=2560000$) put k; as also for $2cc+2dd$ ($=3524$) put l, and for $c^4+d^4-2ccdd$ ($=\overline{cc\backsim dd}|^2=2560000$) put m; and then you'll have from the two foregoing Steps, and the Nature of the Question

	3	$\frac{1}{4}\sqrt{:-x^4+hxx-k:}+\frac{1}{4}\sqrt{:-x^4+lxx-m:}=g.$
3,	4	$\sqrt{:-x^4+hxx-k:}=4g-\sqrt{:-x^4+lxx-m:}$
4⊙2	5	$-x^4+hxx-k=16gg-x^4+lxx-m-8g\sqrt{:-x^4+lxx-m:}$
5,	6	$\frac{l-h}{8g}xx+2g+\frac{k-m}{8g}=\sqrt{:-x^4+lxx-m:}$

Again, for $\frac{h-l}{8g}$ $(=\frac{21}{40})$ put n, and for $2g+\frac{k-m}{8g}$ $(=1560)$ put p; and then you'll have from the 6th Step,

	7	$-nxx+p=\sqrt{:-x^4+lxx-m:}$
7⊙2	8	$nnx^4-2npxx+pp=-x^4+lxx-m.$
8,	9	$:nn+1:\times x^4-:2np+l:\times xx=-m-pp.$

Lastly, for $nn+1$ $(=\frac{2041}{1600})$ put q; for $2np+l$ $(=2\times 2581)$ put $2r$, And for $m+pp$ $(=4993600)$ put s, and then the 9th Step will become

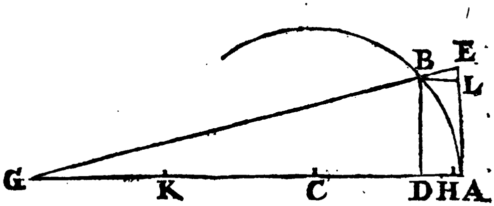
	10	$qx^4-2rxx=-s.$
10 Reduc'd	11	$x=\sqrt{:\frac{1}{q}\times r\pm\sqrt{rr-qs}:}$

$(=\sqrt{:\frac{1600}{2041}\times 2581\pm\sqrt{6661561-6369961}:}=49.46+$ or $=40$; *viz.* two sorts of Trapezia may be found, each of which will answer the Problem.)

PROB. XX.

From Dr. Wallis's *Algebra.*

DA $=v$ the Versed-Sine of the Arc BA of a Circle, and AK $=d$ the Diameter being given: 'Tis required to find the Point G in the Diameter produc'd sufficiently, from which the streight Line GB being drawn and produc'd to E, that the Tangent AE may be nearly equal to the Arc BA.



Solution.

1. Suppose it done, and GBE and GCDA drawn, as also BL ∥ DA,

 2. By

2. By the Property of a Circle $\sqrt{:AD \times DK:} = DB = \sqrt{:dv - vv:}$ which is, by Sir *Isaac Newton*'s Theorem, $= \sqrt{dv} - \frac{\sqrt{v^3}}{2\sqrt{d}} - \frac{\sqrt{v^5}}{8\sqrt{d^3}} - \frac{\sqrt{v^7}}{16\sqrt{d^5}} -$ &c.

3. By the Method of Exhaustions (which will be treated of in Part III. of this Book) the Length of the Arc AB will be found to be $= \sqrt{dv} + \frac{\sqrt{v^3}}{6\sqrt{d}} + \frac{3\sqrt{v^5}}{40\sqrt{d^3}} + \frac{5\sqrt{v^7}}{112\sqrt{d^5}}$ + &c. which is required to be nearly equal to the Tangent AE.

4. The $\triangle$s ELB and EAG are similar; wherefore $EL = AE - DB \cdot\cdot LB = AD :: AE = \text{Arc } AB \cdot\cdot AG = \frac{1}{2}d - \frac{2}{5}v - \frac{12\,vv}{175\,d}$, &c.

5. Whence (First) AG is nearly $= \frac{1}{2}d - \frac{2}{5}v$.

For supposing $AG = \frac{1}{2}d - \frac{2}{5}v$; then $DG = \frac{1}{2}d - \frac{6}{5}v$.

And the $\triangle$s DGB and LBE are similar; wherefore $DG \cdot\cdot DB :: BL = DA \cdot\cdot EL = AE - DB = \frac{2\sqrt{v^3}}{3\sqrt{d}} + \frac{\sqrt{v^5}}{5\sqrt{d^3}} + \frac{23\sqrt{v^7}}{300\sqrt{d^5}}$, &c.

And $AE - DB + DB = AE = \sqrt{dv} + \frac{\sqrt{v^3}}{6\sqrt{d}} + \frac{3\sqrt{v^5}}{40\sqrt{d^3}} + \frac{17\sqrt{v^7}}{1200\sqrt{d^5}}$, &c.

Which is but by $\frac{16\sqrt{v^7}}{525\sqrt{d^5}}$, &c. less than the precedent required Value of AE.

Wherefore taking $AH = \frac{2}{5}DA$, and making $KG = HC$, and then drawing the straight Line GBE, it will cut the Tangent AE in the Point E, so as the said Tangent will be nearly equal to the Arc BA, the Error being only $\frac{16\,v^3}{525\,d^3}\sqrt{dv}$, &c.

But (*Secondly*) if the Point G thus found be not exact enough for your Purpose, taking $AH = \frac{4}{5}DA$, make $7d = 7AK \cdot\cdot \frac{3}{5}v = 3AH :: \frac{4}{5}v = DH \cdot\cdot \frac{12\,vv}{175\,d}$, and you'll have $KG = CH - \frac{12\,vv}{175\,d} = \frac{d}{2} - \frac{2}{5}v - \frac{12\,vv}{175\,d}$, and conse-

consequently, $AG = \frac{1}{2}d - \frac{1}{5}v - \frac{12\,vv}{175\,d}$ which will determine the Point G more accurately than as it has been done before.

For AG being $= \frac{1}{2}d - \frac{1}{5}v - \frac{12\,vv}{175\,d}$, you'll have from the similar Δs DGB and LBE,

$DG \cdot\cdot DB :: BL = DA \cdot\cdot EL = AE - DB = \frac{2\sqrt{v^3}}{3\sqrt{d}} + \frac{\sqrt{v^5}}{5\sqrt{d^3}} + \frac{3\sqrt{v^7}}{28\sqrt{d^5}}$, &c.

And $AE - DB + DB = AE = \sqrt{dv} + \frac{\sqrt{v^3}}{6\sqrt{d}} + \frac{3\sqrt{v^5}}{40\sqrt{d^3}} + \frac{5\sqrt{v^7}}{112\sqrt{d^5}}$, &c. which is so far the same with the required Value of AE, the Operation being not carry'd far enough to discover the Error.

Schol. I.

Hence the Radius $CA = r$, Versed-Sine $DA = v$, and consequently the Sine $BD = s$ of any Arc AB being given, the Length of the Arc $AB = a = AE$ may be nearly found: Thus

The Δs GDB and GAE are similar; wherefore $GD \cdot\cdot DB :: GA \cdot\cdot AE = a = AB$.

That is to say (by the above *Problem*, and because $d = 2r$)

$$3r - \frac{6}{5}v - \frac{12\,vv}{350\,r} \cdot\cdot s :: 3r - \frac{1}{5}v - \frac{12\,vv}{350\,r} \cdot\cdot a \text{ nearly.}$$

Consequently, $p = 15r - 6v - \frac{6\,vv}{35\,r} \cdot\cdot s :: q = p + 5v \cdot\cdot a = \frac{sq}{p}$ nearly.

Now, knowing that the Radius of a Circle being $= r$, its Semi-circumference will be 3.1415926 &c. r, and that each Semi-circle is suppos'd to contain 180 Degrees, I proceed to find how many Degrees in $\frac{sq}{p}$; Thus

The Length of the Semi-circumference, *viz.* 3.1415926 &c. $\times r$ ·· Number of Degrees in the Semi-circumference *viz.* $180° :: \frac{sq}{p}$ ·· Number of Degrees in the Arc AB nearly.

Consequently, Arc $AB = a = \frac{sq}{.01745329 25 \text{ \&c. } pr}$ nearly.

This is the same with Mr. Thomas Wallis's *Theorem which I have seen; and his Method of finding it (which I have not yet seen) is, I presume, the same with the foregoing.*

Scholium

Scholium 2.

Again, Suppofing AG $= \frac{1}{2}d - \frac{2}{5}v = 3r - \frac{2}{5}v = 3 - \frac{2}{5}v$, putting the Radius $r = 1$, as alfo the Number of Degrees in the Arc AB to be given, and the Sine BD $= s$ to be required, it may be found nearly; Thus

Multiply the Number of Minutes in the Arc AB by $\frac{3.1415926 \text{ \&c.}}{180 \times 60}$, *viz.* by .0002908882 &c. (the Length of the Arc of 1 Minute), and call the Product a; Then

The $\triangle$s GDB and GAE are fimilar; wherefore,

$3 - \frac{6}{5}v$ ($=$ GD) $\cdot\cdot$ s (DB) :: $3 - \frac{2}{5}v$ ($=$ GA) $\cdot\cdot$ AE $=$ a nearly; that is (becaufe v is $= 1 - \sqrt{:1 - ss:}$)

$$\frac{9 + 6\sqrt{:1 - ss:}}{5} \cdot\cdot\ s :: \frac{14 + \sqrt{:1 - ss:}}{5} \cdot\cdot\ a$$

Confequently, $9a + 6a\sqrt{:1 - ss:} = 14s + s\sqrt{:1 - ss:}$

By Tranfpofition, $9a - 14s = :s - 6a: \times \sqrt{:1 - ss:}$

By fquaring each Part, $81aa - 252as + 196ss = :ss - 12as + 36aa: \times :1 - ss: = ss - 12as + 36aa - s^4 + 12as^3 - 36aass$.

The laft Step being reduc'd, gives $45aa = 240as - 195ss - 36aass + 12as^3 - s^4$; which is a biquadratick Equation, in which one of the Values of s is $=$ the Sine fought nearly.

This is Mr. Ward's *Theorem for finding nearly the Sine of any given Angle lefs than 45°: But you have much better Methods for that Purpofe in the Beginning of* Part 1.

Note, *What follows is from Mr.* Thomas Wallis.

Again, putting c ($=$ CD) $=$ Σ Arc AB, then $s = \sqrt{:1 - cc:}$, and $v = 1 - c$; wherefore we have, from the fimilar Triangles GDB and GAE, $\frac{9 + 6c}{5}$ (GD) $\cdot\cdot \sqrt{:1 - cc:}$ (DB) :: $\frac{14 + c}{5}$ (GA) $\cdot\cdot$ AE $=$ a nearly.

Confequently, $:9a + 6ca = :14 + c: \times \sqrt{:1 - cc:}$

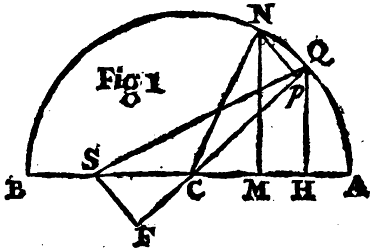
The laft Step, when reduc'd, gives $c^4 + 28c^3 + 195cc + 36aacc - 28c + 108aac = 196 - 81aa$, which is a biquadratick Equation, wherein one of the Values of c is $=$ the Co-fine of AB fought nearly.

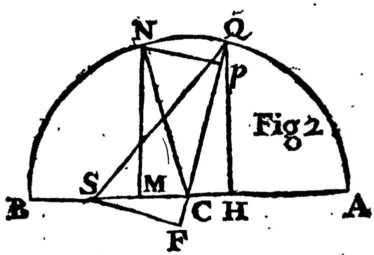
This is, in effect, Mr. Ward's *Theorem for finding nearly the Co-fine of any given Angle lefs than 45°, or (which is the fame thing) the Sine of any Angle greater than 45°.*

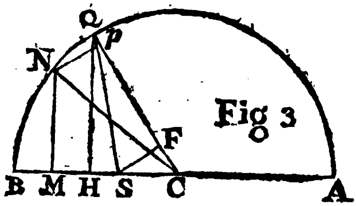
PROB.

PROB. XXI.

From Dr. Keill's *Astronomical Lecture.*







To divide a given Semi-circle, in a given Ratio, by a ſtraight Line drawn from a given Point in the Diameter (near the Center) to its Circumference.

Let C A Q B be a given Semi-circle, C its Center, B A its Diameter, S a Point therein, from which to its Circumference drawing the Line S Q, the Area of B Q S muſt be to that of A Q S in the given Ratio.

Suppoſe the Point ſought Q to be found: Then draw the Line Q C, which produce, if need be, to F, on which let fall the Perpendicular S F.

Divide the Semi-circumference B Q A in N, according to the given Ratio, and, drawing the Line C N, the Areas of the Sectors C N B and C N A will be manifeſtly to each other in

in the given Ratio: And therefore the Area of BQS is $=$ that of the Sector BNC; consequently the Area of the Δ SQC $= \frac{SF \times QC}{2}$ is $=$ Area of the Sector CNQ $= \frac{NQ \times QC}{2}$; wherefore SF is $=$ Arch NQ.

Draw $Np \perp CQ$, and NM and QH $\perp$s BA.

Now let the known Quantities, *viz.* the Radius CQ be $= 1$, Sine of the Arch AN $=$ NM $= s$, its Co-sine CM $= cs$, and SC $= b$; And suppose the unknown Arch NQ $=$ SF $= a = ?$

Then (by the Method of Exhaustions) the Sine of the Arc NQ $=$ Np will be found $= a - \frac{a^3}{6} + \frac{a^5}{120}$, &c. and its Co-sine pC $= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720}$, &c. Now, in *Fig.* $\begin{smallmatrix}1\\2,\ 3,\end{smallmatrix}$ by *Theorem* $\begin{smallmatrix}1\\9,\ 3\end{smallmatrix}$ in the *Lemma* to *Problem* 12. $NM \times pC \mp Np \times MC$ [Radius being 1] $=$ Sine of the Arc QA $=$ QH; that is $s \mp csa - \frac{sa^2}{2} \pm \frac{csa^3}{6} + \frac{sa^4}{24} \mp \frac{csa^5}{120}$, &c. $=$ QH. The Δs CQH and CSF are similar; Wherefore 1 (CQ) $\cdot\cdot$ b (CS) $::$ QH $\cdot\cdot$ SF $= a = bs \mp bcsa - \frac{bsa^2}{2} \pm \frac{bcsa^3}{6} + \frac{bsa^4}{24} \mp \frac{bcsa^5}{120}$, &c. Consequently, by Transposition, Division, and putting $\frac{1}{bs} \pm c = n$, we have $1 = na + \frac{aa}{2} \mp \frac{ca^3}{6} - \frac{a^4}{24} \pm \frac{ca^5}{120}$, &c. where Note, b being much less than any of the other given Quantities, that n will be much more than any of them. Now in order to determine the Value of a, express'd in and by this Series, if we suppose it, *viz.* $a = \frac{A}{n} + \frac{B}{nn} + \frac{C}{n^3} + \frac{D}{n^4} + \frac{E}{n^5} + \frac{F}{n^6} +$ &c. then A, by the following Method, will presently be seen to be $= 1$, and B $= 0$: Suppose therefore $a = \frac{1}{n} * + \frac{C}{n^3} + \frac{D}{n^4} + \frac{E}{n^5} + \frac{F}{n^6}$, &c. Then

$$\left.\begin{array}{llllll}
na = 1 & * + \frac{C}{n^2} & + \frac{D}{n^3} & + \frac{E}{n^4} & + \frac{F}{n^5}, & \&c. \\
+\frac{aa}{2} = & + \frac{1}{2n^2} & * & + \frac{C}{n^4} & + \frac{D}{n^5}, & \&c. \\
\mp\frac{ca^3}{6} = & & \mp \frac{c}{6n^3} & * & \mp \frac{cC}{2n^5}, & \&c. \\
-\frac{a^4}{24} = & & & -\frac{1}{24n^4} & *\ , & \&c. \\
\pm\frac{ca^5}{120} = & & & & \pm\frac{c}{120n^5}, & \&c. \\
\&c. = & & & & \&c. &
\end{array}\right\} = 1.$$

Now by equating the refpective Terms, we have

$$1 = 1.$$

Alfo $\frac{C}{n^2} + \frac{1}{2n^2} = 0$; therefore $C = -\frac{1}{2}$.

Again $\frac{D}{n^3} \mp \frac{C}{6n^3} = 0$; confeq. $D = \pm \frac{c}{6}$

Alfo $\frac{E}{n^4} + \frac{C}{n^4} - \frac{1}{24n^4} = 0$; wherefore $E = \frac{13}{24}$

In like Manner F will be found $= \mp \frac{51c}{120}$.

&c.

Confequently $a = \frac{1}{n} * - \frac{1}{2n^3} \pm \frac{c}{6n^4} + \frac{13}{24n^5} \mp \frac{51c}{120n^6}$ &c. *Sine fine.*

This Series expreffes the Length of the Arch $NQ = a$: But to have it in Degrees, fay, as 6.28318 &c. the Length of the Circumference, is to 360, the Number of Degrees in the Circumference, fo is this Series, the Length of the Arch $NQ = a$, to the Number of Degrees in the faid Series: Confequently if $\frac{360}{6.28318\,\&c.} = 57.29578$ be put $= R$, we fhall have the Arch

$$a \text{ in Degrees} = \frac{R}{n} - \frac{R}{2n^3} \pm \frac{cR}{6n^4} + \frac{13R}{24n^5} \mp \frac{51cR}{120n^6}, \ \&c.$$

Now in all the Planets elliptical Orbits, the Eccentricity S C is very fmall, in refpect of the mean Diftance C Q; and therefore this Series is of good Ufe in determining their true Anomalies, their Eccentricities being given, &c. For Inftance,

In the Earth's Orbit the Eccentricity SC is .01691, when the mean Distance CQ is 1. Suppose we are to find the true Anomaly, *viz.* the Arc AQ, when the mean Anomaly AN is 30 Degrees.

Here $b = .01691$, $s = .5$ (the Sine of 30°.) wherefore $\frac{1}{bs} = 118.2732$: Also $cs = \sqrt{.75}$; wherefore $\frac{cs}{s} = c = \sqrt{\frac{.75}{.25}} = \sqrt{3} = 1.7320 +$: consequently $n = 120.0052 +$

$$LR = L\,57.29578 = 1.7581226$$
$$Ln = L\,120.0052 = 2.0792000$$

$$L\frac{R}{n} = \bar{1}.6789226 \text{ Number } .4774441$$

$$LR = 1.7581226$$
$$L\,2n^3 = L\,2 + 3Ln = 6.5386300$$

$$L\frac{R}{2n^3} = \bar{5}.2194926 \text{ Number } .0000165$$

$$\frac{R}{n} - \frac{R}{2n^3} = .477427 \text{ Deg.}$$

$$\begin{array}{r|l} & 60 \\ \hline 28 & 645620 \\ & 60 \\ \hline 38 & 737200 \\ & 60 \\ \hline 44 & 232000 \end{array}$$

That is to say, the Arch $QN = a$ is $= .477427$ Degrees, or 00° 28′ 38″ 44‴ +; Consequently 30° (AN) − 28′ 38″ 44‴ + (NQ) = 29° 31′ 21″ 15‴ + = AQ sought.

PART III.

Of Exhaustions.

THE Method of EXHAUSTIONS is founded upon the following

Lemma.

If by adding or subtracting a Quantity, be it ever so small, (*viz.* an indefinitely little Quantity) to or from any determinate Quantity, the Sum, or Remainder be accordingly greater or less than another Quantity; I say the two last mentioned Quantities must be equal to each other.

Exposition.

Suppose a to be an indefinitely small Quantity, and $b =$ any determinate Quantity: Now if $b + a$ be $\sqsubset x$

And $b - a \sqsupset x$; I say x is $= b$.

Demonstration.

For, if you suppose that either of them, as b, cou'd be $\sqsubset$ the other x, it must be by some Quantity, which suppose $= d$; then b wou'd be $= x + d$, and the Consequence wou'd be, that d is $\sqsupset a$.

But altho' d had been less than $b \times \dfrac{1}{999999, \text{\&c. till there are Octillions of 9's writ one after another}}$, or less than $b \times$ the Square of that Number, or less than $b \times$ its Cube, &c. yet $b \times$ some Power of the said Number must be $\sqsupset d$, by Post. Lib. 10. *Eucl. El.* which Power $\times\, b$ let a (for 'tis evident that it may) be equal to or less than; then a will be $\sqsupset d$, which is contrary to the foregoing Consequence: And therefore, by supposing that b is $\sqsubset x$ a Contradiction ensues, which proves that b is not $\sqsubset x$; And for the same Reasons x is not $\sqsubset b$. Consequently x is $= b$. *Q. E. D.*

That Part of Exhaustions, which treats of Quadratures, is likewise founded upon the following *Lemma*; that is, upon its *Scholia* and *Corollaries*.

Lemma II.

If n be $=$ an indefinite Number, and $p =$ such a Number, that the Sum (s) of 1^p, 2^p, 3^p, 4^p, &c. continued to n Terms be $= \dfrac{n^{p+1}}{p+1}$ ϕ, suppose $= \dfrac{n^{p+1}}{p+1} + u \times n^p \phi$; then I say the Sum ($z$) of 1^{p+1}, 2^{p+1}, 3^{p+1}, 4^{p+1}, &c. continued to n Terms will be $= \dfrac{n^{p+2} + :pu+1: \times n^{p+1}}{p+2}$ ϕ.

Demonstration.

It is evident that z is $=$

$$\left.\begin{array}{l} 1^p + 2^p + 3^p + \&c. + n^p \\ \quad\;\, + 2^p + 3^p + \&c. + n^p \\ \qquad\quad\;\; 3^p + \&c. + n^p \\ \qquad\qquad\quad\; + \&c. + n^p \end{array}\right\} \begin{array}{c}\&c. \text{ to } n \text{ Ranks} \\ \text{continued.}\end{array} = \left.\begin{array}{l} 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \end{array}\right\} \begin{array}{c}\&c. \text{ to } n \text{ Ranks} \\ \text{Continued.}\end{array} - \left.\begin{array}{l} 0 \\ 1^p \\ 1^p + 2^p \\ 1^p + 2^p + 3^p \end{array}\right\} \begin{array}{c}\&c. \text{ to } n \text{ Ranks} \\ \text{continued.}\end{array}$$

No

Now if s be $= \frac{n^{p+1}}{p+1} + u \times n^p \phi$, then the Sum of the foregoing greater Series is $= \frac{n^{p+2}}{p+1} + u \times n^{p+1} \phi$.

Again, in order to find the Sum of the lesser Series; consider that since o is the first Rank thereof, 1^p the 2d, $1^p + 2^p$ the 3d &c. that $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-1}|^p$ will be $=$ the nth Rank, Also $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-2}|^p =$ the $\overline{n-1}$th Rank, also $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-3}|^p =$ the $\overline{n-2}$th Rank, &c. of the lesser Series: Wherefore, if s be $= \frac{n^{p+1}}{p+1} + u \times n^p \phi$, the said nth Rank will be $= \frac{\overline{n-1}|^{p+1}}{p+1} + u \times \overline{n-1}|^p \phi$; Also the said $\overline{n-1}$th Rank will be $= \frac{\overline{n-2}|^{p+1}}{p+1} + u \times \overline{n-2}|^p \phi$; Also the said $\overline{n-2}$th Rank $= \frac{\overline{n-3}|^{p+1}}{p+1} + u \times \overline{n-3}|^p \phi$; &c. Wheref. the Sum of the said lesser Series is $= : p+1 :) : \overline{n-1}|^{p+1} + \overline{n-2}|^{p+1} + \overline{n-3}|^{p+1} +$ &c. to $n-1$ Terms continued $: + u \times : \overline{n-1}|^p + \overline{n-2}|^p + \overline{n-3}|^p +$ &c. to $\overline{n-1}$ Terms continued $: \phi :$ But $: p+1 :) : \overline{n-1}|^{p+1} + \overline{n-2}|^{p+1} + \overline{n-3}|^{p+1} +$ &c: is manifestly $= \frac{\overline{z-n}|^{p+1}}{p+1}$: And, if $u \times : \overline{n-1}|^p + \overline{n-2}|^p + \overline{n-3}|^p +$ &c: be $= u \times \frac{\overline{n-1}|^{p+1}}{p+1} \phi$; then the Difference of the Greater and Lesser foregoing Series is $= \frac{n^{p+2}}{p+1} + u \times n^{p+1} \phi - \frac{\overline{z-n}|^{p+1}}{p+1} - u \times \frac{\overline{n-1}|^{p+1}}{p+1} \phi$, which, by Supposition, is $= z = \frac{n^{p+2} + \overline{pu+u} \times n^{p+1} - z + n^{p+1} - u \times n^{p+1} \phi}{p+1}$ (for $\overline{n-1}|^{p+1}$ is $= n^{p+1} \phi$); therefore $n^{p+2} + n^{p+1} + pu \times n^{p+1} \phi - z = pz + z$; conseq. $z = \frac{n^{p+2} + \overline{pu+1} \times n^{p+1} \phi}{p+2}$

Q. E. D.

Schol.

Scholium I.

If n be $=$ an indefinite Number, and $p = 1$; then $1^p + 2^p + 3^p +$ &c. $+ n^p$ is (by what has been said in ÷) $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p$; consequently $1^{p+1} + 2^{p+1} + 3^{p+1} +$ &c. $+ n^{p+1}$ will be, by our *Lem.* 2. $= \frac{n^{p+2}}{p+2} + \frac{1}{2}n^{p+1}\phi$ ($= \frac{n^{p+2}}{p+2} + \frac{\frac{1}{2}p+1}{p+2} \times n^{p+1}\phi$). Again, if p be $= 2$, then $1^p + 2^p + 3^p$ + &c. $+ n^p$ (by this *Schol.*) is $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi$; therefore $1^{p+1} + 2^{p+1} + 3^{p+1} +$ &c. $+ n^{p+1}$ will be (by our *Lem.* 2.) $= \frac{n^{p+2}}{p+2} + \frac{1}{2}n^{p+1}\phi$; &c.

Corollary I.

Wherefore if n be $=$ an indefinite Number, and $p =$ any affirmative whole Number $\sqsubset 1$, $1^p + 2^p + 3^p +$ &c. $+ n^p$ will be $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi$: And conseqently $1^p + 2^p + 3^p$ + &c. $+ \overline{n-1}^p = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi - n^p = \frac{n^{p+1}}{p+1} - \frac{1}{2}n^p\phi$

Scholium II.

1. Again, by *Scholium* to *Prob.* 2. *Chap.* 3. of this Part III. it is further demonstrated, that n being $=$ an indefinite Number, and $p =$ any affirmative whole Number, $1^{\frac{1}{p}} + 2^{\frac{1}{p}} + 3^{\frac{1}{p}} +$ &c. $+ n^{\frac{1}{p}}$ is $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1}\ominus$; therefore $1^{\frac{1}{p}+1} + 2^{\frac{1}{p}+1} + 3^{\frac{1}{p}+1} +$ &c. $+ n^{\frac{1}{p}+1}$ is, by our *Lem.* 2. $= \frac{n^{\frac{1}{p}+2}}{\frac{1}{p}+2}\ominus$; &c.

2. Likewise, by the said *Scholium* to *Prob.* 2. *Chap.* 3. of this Part III. it is demonstrated that $1^{\frac{1}{p}} + 2^{\frac{1}{p}} + 3^{\frac{1}{p}}$ + &c.

continued

continued to $n-1$ Terms is $= \dfrac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1}\,\Theta$; Let us suppose it $= \dfrac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} - x \times n^{\frac{1}{p}}\,\phi$; then this Series continued to n Terms is $= \dfrac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} + \overline{1-x} \times n^{\frac{1}{p}}\,\phi$: And $1^{\frac{1}{p}+1} + 2^{\frac{1}{p}+1} + 3^{\frac{1}{p}+1}$ + &c. continued to n Terms, will be (by our *Lemma* 2.) $= \dfrac{n^{\frac{1}{p}+2} + \overline{\frac{1-x}{p}+1} \times n^{\frac{1}{p}+1}\,\phi}{\frac{1}{p}+2}$; consequently this last Series continued only to $n-1$ Terms, is $= \dfrac{n^{\frac{1}{p}+2} - :\frac{1}{p}x+1: \times n^{\frac{1}{p}+1}\,\phi}{\frac{1}{p}+2} = \dfrac{n^{\frac{1}{p}+2}}{\frac{1}{p}+2}\,\Theta$. &c.

From what hath been said, it is manifest that q being likewise $=$ any affirmative whole Number, $1^{\frac{1}{p}+q} + 2^{\frac{1}{p}+q} + 3^{\frac{1}{p}+q}$ + &c. $= 1^{\frac{1+pq}{p}} + 2^{\frac{1+pq}{p}} + 3^{\frac{1+pq}{p}}$ + &c. continued to $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}$ Terms is $= \dfrac{n^{\frac{1+pq}{p}+1}}{\frac{1+pq}{p}+1}\left\{\begin{matrix} \Theta \\ \Theta \end{matrix}\right\}$: Wherefore it may be demonstrated, by the like Method with that us'd in *Probl.* 2. and its *Schol.* in *Chap.* 3. of this Part, that $1^{\frac{p}{1+pq}} + 2^{\frac{p}{1+pq}} + 3^{\frac{p}{1+pq}}$ + &c. continued to $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}$ Terms is $= \dfrac{n^{\frac{p}{1+pq}+1}}{\frac{p}{1+pq}+1}\left\{\begin{matrix} \Theta \\ \Theta \end{matrix}\right\}$.

Now tho' p and q be limitted so as to be equal to whole Numbers, yet there is no Number, either whole or fracted, but $\dfrac{p}{1+pq}$ or $\dfrac{1+pq}{p}$ may be equal to: Therefore

Corol-

Corollary II.

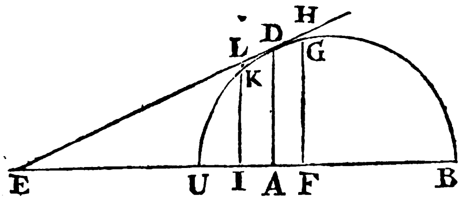
If m be $=$ any affirmative determinate Number whatsoever, and $n =$ an indefinite Number, $1^m + 2^m + 3^m + 4^m +$ &c. continued to n, and $: n - 1 :$ Terms will be equal to $\frac{n^{m+1}}{m+1}$ • and $\frac{n^{m+1}}{m+1}$ • respectively.

CHAP. I.

How to draw Tangents to Curve-Figures.

PROP. I.

To draw a Tangent to a Circle.



LET the Diameter VB of the Circle $= 2r$, VA $= x$, then AB $= 2r - x$ } Be given.

$$\left.\begin{array}{r} VB = 2r \\ VA = x \\ AB = 2r - x \end{array}\right\} \text{Be given.}$$

'Tis required to find the Sub-Tangent A E $= t = ?$

Draw AD $\perp$ UB; draw also the Tangent ED, which produce to H, and imagine the Lines HGF and LKI, intersecting the Tangent produc'd in H and L, the Arc in G and K, and the Diameter in F and I, to be drawn Parallels to, and at an equal indefinitely small Distance from DA, and suppose that Distance to be $= a =$ AF $=$ AI.

Solution.

By the Property of a Circle BA $\times$ VA $=$ DAq $= : 2r - x : \times x = 2rx - xx$; also BF $\times$ UF $=$ FGq, and BI $\times$ UI $=$ IKq; that is,

$: 2r - x \mp a : \times : x \pm a : = 2rx - xx \pm 2ra \mp 2xa - aa = \frac{FGq}{IKq}$

The $\triangle$s EDA, EHF and ELI are similar; wherefore

$EAq \cdot\cdot ADq :: \begin{cases} EFq \cdot\cdot FHq \\ EIq \cdot\cdot ILq \end{cases}$

That is $tt \cdot\cdot 2rx - xx :: tt \pm 2ta + aa \cdot\cdot \frac{2rx - xx \pm 4rxta \mp 2xxta + 2rxaa - xxaa}{tt} = \frac{FHq}{ILq}$

But FHq is $\sqsubset FGq$, and $ILq \sqsubset IKq$ (let a be ever so small, unless it be equal to, or less than Nothing, as it is not by Supposition); that is, $2rx - xx \frac{\pm 4rxta \mp 2xxta + 2rxaa - xxaa}{tt} \sqsubset 2rx - xx \pm 2ra \mp 2xa - aa.$

And, by Abbreviat. Divis. Multiplicat. and Transposit. $\pm 4rxt \mp 2xxt + 2rxa + tta - xxa \sqsubset \pm 2rtt \mp 2xtt.$

That is to say, $+ 4rxt - 2xxt + : 2rx + tt - xx : \times a \sqsubset + 2rtt - 2xtt.$

And $- 4rxt + 2xxt + : 2rx + tt - xx : \times a \sqsubset - 2rtt + 2xtt$;

Consequently, (from the last Step) $4rxt - 2xxt - : 2rx + tt - xx : \times a \sqsupset + 2rtt - 2xtt.$

From the last, and last but two Steps, it is manifest that any Quantity, be it ever so small, so that it is more than Nothing, (viz. $2rx + tt - xx : \times a$) being taken from or added to $4rxt - 2xxt$ gives the Remainder or Sum accordingly less or greater than $2rtt - 2xtt$; consequently (by our * *Lemma* 1.) $4rxt - 2xxt = 2rtt - 2xtt$. Whence $\frac{2rx - xx}{r - x} = t$.

* *Lema* 1. *to Exhaustions.*

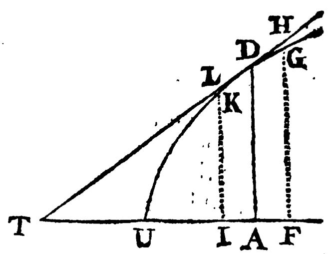
PROP. II.

To draw a Tangent to a Parabola.

Let the Abscissa UA $= x$, and the Parameter $= p$ } Be given.

'Tis required to find the Subtangent AT $= t = ?$

Draw AD an Ordinate to UA; draw also the Tangent TD, which produc'd to H, and imagine the Lines HGF



and L K I, intersecting the Tangent produc'd in H and L, the Curve in G and K, and the Absciss produc'd in F and I, to be drawn ||s to, and at an equal indefinitely small Distance from D A; and suppose that Distance to be $= a =$ A F $=$ A I.

Solution.

By the Property of a Parabola $px = ADq$; as also $px \pm pa = \frac{GFq}{KIq}$

The Δs TAD, TFH and TIL are similar; therefore

$$TAq \cdot\cdot ADq :: \begin{cases} TFq \cdot\cdot FHq \\ TIq \cdot\cdot ILq; \end{cases}$$

That is $tt \cdot\cdot px :: tt \pm 2ta + aa \cdot\cdot px \frac{\pm 2pxta + pxaa}{tt} = \frac{FHq}{ILq}$

But HFq is $\sqsubset GFq$, and $LIq \sqsubset KIq$; that is $px \frac{\pm 2pxta + pxaa}{tt} \sqsubset px \pm pa$; wherefore

$\pm 2xt + xa \sqsubset \pm tt$; that is to say,

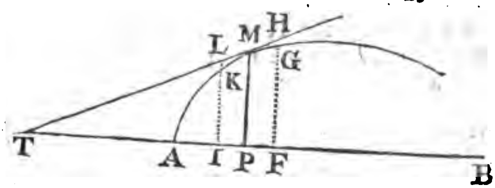
$+ 2xt + xa \sqsubset + tt$.

And $- 2xt + xa \sqsubset - tt \therefore + 2xt - xa \sqsupset + tt$.

Now from the last, and last but two Steps, $2xt$ is (by our *Lemma* 1.) $= tt$: Consequently, $2x = t$.

PROP. III.

To draw a Tangent to an Ellipsis.



Let the Transverse-Diameter A B $= b$ }
The Parameter $= p$ } Be given.
A P $= x$ }
Then P B $= b - x$

'Tis required to find the Sub-Tangent P T $= t = ?$

Draw P M an ordinate to A B; draw also the Tangent TM, which produce to H, and imagine the Lines H G F and L K I, intersecting the Tangent produc'd in H and L, the Curve in G and K, and the transverse Diameter in F and I, to be drawn

at

at an indefinitely-little equal Distance from, and parallel to MP; and suppose the indefinitely short Line IP $=$ PF $= a$.

Solution.

By the Property of an Ellipse $x \times : b - x :: \mathrm{PM}q :: b \,..\, p$; consequently $\mathrm{PM}q = \dfrac{pbx - pxx}{b}$

In like manner will be found $\dfrac{:x \pm a: \times :b - x \mp a: \times p}{b} =$

$\begin{matrix}\mathrm{FG}q \\ \mathrm{IK}q\end{matrix} = \dfrac{pbx - pxx \mp 2pxa \pm pba - paa}{b}$.

The Δs TPM, TFH, TIL are similar; wherefore tt $(\mathrm{TP}q) \,..\, \dfrac{pbx - pxx}{b}\, (\mathrm{PM}q) :: tt \pm 2ta + aa \left(\begin{matrix}\mathrm{TF}q \\ \mathrm{TI}q\end{matrix}\right)$ $..\, \begin{matrix}\mathrm{FH}q \\ \mathrm{IL}q\end{matrix} = \dfrac{pbxtt - pxxtt \pm 2pbxta \mp 2pxxta + pbxaa - pxxaa}{ttb}$.

But FHq is $\sqsubset$ FGq, and ILq $\sqsubset$ IKq; that is $\dfrac{pbxtt - pxxtt \pm 2pbxta \mp 2pxxta + pbxaa - pxxaa}{ttb} \sqsubset \dfrac{pbx - pxx \mp 2pxa \pm bpa - paa}{b}$;

which Step, after due Reduction, gives $\pm 2bx \mp 2xx + \dfrac{bx + tt - xx}{t}\, a \sqsubset \pm tb \mp 2tx$.

That is to say, $+ 2bx - 2xx \ominus \sqsubset + tb - 2tx$.

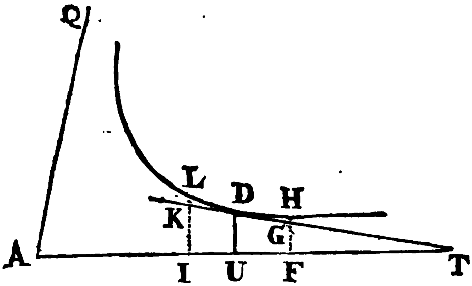
And $- 2bx + 2xx \ominus \sqsubset - tb + 2tx$ ∴

$+ 2bx - 2xx \ominus \sqsupset + tb - 2tx$.

From the last, and last but two Steps, $2bx - 2xx$ is (by our *Lemma* 1.) $= tb - 2tx$: Whence $\dfrac{2bx - 2xx}{b - 2x} = t$.

PROP. IV.

To draw a Tangent from the Asymptotes of an Hyperbola to its Curve.



Let A Q and A T represent the Asymptotes of the Hyperbola L D H. Suppose AU $= x$, and the Property of the Hyperbola to be given; 'tis required to find the Sub-tangent UT $= t = ?$

Note, UD being drawn || A Q, one of the Asymptotes, the Property of the Apollonian Hyperbola, in relation to its Asymptotes, is $pp =$ AU × UD, where p represents a determinate Quantity, and consequently may be expressed by unity; so that the same Equation may become $1 =$ AU × UD; that is, $1 = x \times UD$, or $x = \frac{1}{UD} = \overline{UD}|^{-1}$. Whence all sorts of Hyperbolas may be set forth by this general Property AU $= x = \overline{UD}|^{-m}$. Then

Draw the Tangent TD, which produce to K, and imagine the Lines LKI and HGF, intersecting the Curve in L and H, the Tangent produc'd in K and G, and the Asymptote AT in I and F, to be drawn at an a, or indefinitely little equal Distance from and || DU. Then

Solution.

UA $= x$ being (by Supposition) $= \overline{UD}|^{-m}$, $x^{-\frac{1}{m}}$ is therefore $=$ UD;

Also AI $= x - a = \overline{IL}|^{-m}$ (by the Property of the *Fig.*); therefore $\overline{x-a}|^{-\frac{1}{m}} =$ IL: And, for the like Reason, $\overline{x+a}|^{-\frac{1}{m}} =$ FH.

The $\triangle$s TDU, TKI and TGF are similar; wherefore $t : x^{-\frac{1}{m}} :: t \pm a : 1 \pm \frac{a}{t} \times x^{-\frac{1}{m}} = \frac{IK}{FG}$: But $\frac{IL}{FH} =$

IK

$\frac{IK}{FG}$; that is $\overline{x \mp a}|^{-\frac{1}{m}} \sqsubset : 1 \pm \frac{a}{t} : \times x^{-\frac{1}{m}}$; therefore (by raising each Part to the mth Power) $\overline{x \mp a}|^{-1} \left(= \frac{1}{x \mp a} \right) = \frac{1}{x} \pm \frac{a}{xx} + \frac{aa}{xxx} \phi$ is $\sqsubset : 1 \pm m \times \frac{a}{t} + \frac{mm - m}{2} \times \frac{aa}{tt} \phi : \times \frac{1}{x}$; which Step, when reduc'd, gives $\pm \frac{1}{x} + \frac{a}{xx} - \frac{mm - m}{2tt} a \phi \sqsubset \frac{m}{t}$; that is to say (since $\frac{a}{xx} - \frac{mm - m}{2tt} a \phi$ is $= \ominus$, as it is manifestly, by the preceding Step) $+ \frac{1}{x} \ominus \sqsubset + \frac{m}{t}$, and $- \frac{1}{x} \ominus \sqsubset - \frac{m}{t} \therefore + \frac{1}{x} \ominus \sqsupset + \frac{m}{t}$. From the last, and last but two Steps, $\frac{1}{x}$ is (by our *Lemma* 1.) $= \frac{m}{t} \because t = mx$.

N. B. *In these Operations, be sure to take Care that your Multiplicator or Divisor be always* $\sqsubset$ o; *otherwise the Majority will not hold, as in the foregoing Step: Thus, if* h *be* $\sqsubset$ k, *and you multiply each by* l, *if* l *be* $\sqsubset$ o, *then* lh *will be* $\sqsubset$ lk; *but, if* l *be* $\sqsupset$ o, *then* lh *will be* $\sqsupset$ lk.

CHAP.

CHAP. II.

De Maximis & Minimis.

PROP. I.

To divide the given Quantity b, into two such Parts, that their Product may be a *Maximum*.

Suppose $x =$ one Part, then $b - x =$ to the other Part of b, and then $: b - x : \times x = b x - x x =$ *Maximum*.

Solution.

Suppose the Part x to be $\genfrac{}{}{0pt}{}{\text{increased}}{\text{diminish'd}}$ by the indefinitely small, or less than any assignable Quantity a, then the other Part of b will be $b - x \mp a$; And $: x \pm a : \times : b - x \mp a : = b x - x x \pm b a \mp 2 x a - a a$.

But, because $b x - x x$ is (by Supposition) a *Maximum*; therefore it is $\sqsubset b x - x x \pm b a \mp 2 x a - a a$: Whence $o \sqsubset \pm b a \mp 2 x a - a a$; and $\pm 2 x \mp a \sqsubset \pm b$; that is to say,

$$+ 2 x + a \sqsubset + b$$

and $- 2 x + a \sqsubset - b \therefore + 2 x - a \sqsupset + b$.

Now it appears by the last, and the last but two Steps, that let a be ever so small a Quantity, provided it be more than Nothing, if it be subtracted from $2 x$, the Remainder will be less than b, but if it be added to $2 x$, the Sum will be greater than b; consequently, by our *Lemma* 1. $2 x = b$; and $x = \frac{1}{2} b$.

PROP. II.

To divide the given Quantity b into three such Parts, as, they being multiplied together, shall produce a *Maximum*.

Suppose $b - x$ to be one of the Parts required, then x will be $=$ the Sum of the other two Parts: But the greatest Product that can be made of any two Parts of x is, by the foregoing *Proposition*, $\frac{x}{2} \times \frac{x}{2}$: The Question propos'd is therefore reduc'd to this, *viz.* $\frac{x}{2} \times \frac{x}{2} \times : b - x : = \frac{b x x - x^3}{4} =$ *Maximum*. Quere x, it being $=$ an affirmative Quantity.

Suppose

Suppose x to be $\begin{matrix}\text{increas'd}\\ \text{diminish'd}\end{matrix}$ by the indefinitely little Quantity a, then $\frac{bxx - xxx}{4}$ will become $\frac{x \pm a}{2} \times \frac{x \pm a}{2} \times : b - x \mp a :$; But the former being a *Maximum*, is therefore greater than the latter; that is $\frac{bxx - x^3}{4} \sqsubset \frac{bxx - x^3 \pm 2bxa \mp 3x^2a + baa - 3xaa \mp a^3}{4}$; wherefore $0 \sqsubset \frac{\pm 2bxa \mp 3xxa + baa - 3xaa \mp a^3}{4}$: And, by multiplying each Part by $\frac{4}{a}$, and transposing, $\pm 3xx + 3xa - ba\phi \sqsubset \pm 2bx$.

Hence $3xa - ba\phi = 0$; And
$+ 3xx\ 0 \sqsubset 2bx$
and $- 3xx\ 0 \sqsubset - 2bx \therefore 3xx\ 0 \sqsubset 2bx$; consequently (from the last, and last but two Steps) $3xx = 2bx$, by our *Lemma* 1.; Wherefore $x = \frac{2}{3}b$; therefore one and each of the Parts required $= (b - x =)\ b - \frac{2}{3}b = \frac{1}{3}b$.

PROP III.

If n and m be known Quantities, and $n \sqsubset m$, and if $x^m - x^n = $ *Maximum*: It is required to find the Value of x, it being affirmative.

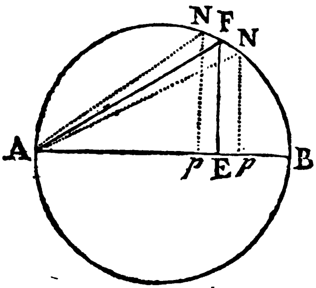
Suppose x to be $\begin{matrix}\text{augmented}\\ \text{decreased}\end{matrix}$ by the indefinitely-little Quantity a; then $x^m - x^n$ will become $\overline{x \pm a}|^m - \overline{x \pm a}|^n$: But the former being a *Maximum*, is therefore greater than the latter; that is (by Sir *Is. Newton*'s Theorem) $x^m - x^n \sqsubset x^m \pm mx^{m-1}a + m \times \frac{m-1}{2}x^{m-2}a^2\phi - x^n \mp nx^{n-1}a - \frac{nn-n}{2}x^{n-2}aa\phi$: Wherefore $0 \sqsubset \pm mx^{m-1}a + \frac{mm-m}{2}x^{m-2}a^2 \mp nx^{n-1}a - \frac{nn-n}{2}x^{n-2}aa\phi$: And (by dividing each Part by a, and transposing) $\pm nx^{n-1} + \frac{nn-n}{2}x^{n-2}a - \frac{mm-m}{2}x^{m-2}a\phi \sqsubset \pm mx^{m-1}$. By

By this Step it is manifest that $\frac{nn-n}{2}x^{n-2}a - \frac{mm-m}{2}x^{m-2}a$ is $= o$.

And $\because +nx^{n-1} \not\sqsubset +mx^{m-1}$
and $-nx^{n-1} \not\sqsubset -mx^{m-1} \therefore +nx^{n-1} \not\sqsupset +mx^{m-1}$

Consequently (from the last, and last but two Steps) nx^{n-1} is (by our *Lemma* 1.) $= mx^{m-1}$. Whence $x = \left(\frac{m}{n}\right)^{\frac{1}{n-m}}$.

PROP. IV.



Of all the Cones that can be inscribed in a given Sphere; 'tis required to find that which has the greatest convex Surface.

This *Proposition* amounts to no more than this;

To determine the Point E in the Diameter A B, of the Circle A B F, so that the Rectangle, comprehended under A F and F E, be the greatest of all the like Rectangles comprehended under A N and N p.

For if we imagine the Semi-circle A F B to revolve about its Axis A B, it is evident that the Semi-circle describes a Sphere, and the ∟d △s A F E and A N p describe Cones inscribed in the same Sphere, whose Surfaces are proportional to the respective Rectangles A F × F E, and A N × N p.

Suppose the Diameter of the Sphere, *viz.* A B $= 2r$, the unknown Quantity A E $= x =$? And the indefinitely short Line pE $=$ Ep $= a$:

Then (by the Property of a Circle) $\sqrt{:2rx - xx:} =$ E F, and $\sqrt{:2rx - xx + xx:} = \sqrt{2rx} =$ F A; and $\sqrt{2rx} \times \sqrt{:2rx - xx:} = \sqrt{:4rrxx - 2rx^3:} =$ *Maximum.* Quere x, it being $=$ an affirmative Quantity.

Suppose x (A E) to be $\begin{matrix}\text{increas'd}\\ \text{diminish'd}\end{matrix}$ by the indefinitely little Quantity a (pE); then $\sqrt{:4rrxx - 2rx^3:}$ ($=$ AF × F E) will become $\sqrt{:4rrxx - 2rx^3 \pm 8rrxa \mp 6rxxa + 4rraa - 6rxaa \mp 2ra^3:}$ ($=$ A N × pN); But the former being a *Maximum*, is therefore greater than the latter, and

and consequently (each of them being more than Nothing) the Square of the former, *viz.* $4rrxx - 2rx^3$ is $\sqsubset$ the Square of the latter, *viz.* $4rrxx - 2rx^3 \pm 8rrxa \mp 6rxxa + 4rra^2 - 6rxa^2 \mp 2ra^3$:

Wherefore $\pm 6rxx + 6rxa - 4rra\circ \sqsubset \pm 8rrx$.

Hence $6rxa - 4rra\circ = \circ$

As also $\because + 6rx^2\circ \sqsubset + 8rrx$

and $-6rxx\circ \sqsubset - 8rrx \because + 6rxx\circ \sqsupset + 8rrx$,

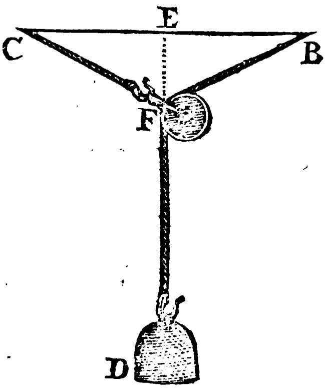
And consequently,

$6rxx$ is $\doteq 8rrx$, by our *Lemma* 1.

Whence $x = \frac{4}{3}r$.

PROP. V.

Let CF be a Chord, whose Length must not exceed CB, and let one End of the Chord be made fast at C, and to the other End fasten the Pulley F, about which suspend the Weight D, by the Chord DFB, which must be of a sufficient Length, fastening one End thereof at B, and let the Points B and C be in the same horizontal Line BC: And suppose both the Chords and the Pulley to be without Weight; 'tis required to find the lowest Descent of the Weight D.



The Chords and the Pulley being without Weight, or the Weight D sufficiently heavy, it is evident that the Weight D will descend below the horizontal Line CB, as low as the Chords CF and BFD will permit; and therefore DFE will represent the greatest Descent of the Weight D.

Suppose the known Quantities DFB $= b$, CB $= c$, and CF $= d$: And suppose the unknown and variable Quantity CE $= x$; then is EF $= \sqrt{:dd - xx:}$, and FB $= \sqrt{:dd + cc - 2cx:}$, and DFE $= b - \sqrt{:dd + cc - 2cx:} + \sqrt{:dd - xx:} =$ *Maximum*. Quere x.

Suppose x to be $\begin{matrix}\text{augmented}\\\text{diminished}\end{matrix}$ by the indefinitely small Quantity a, then $b - \sqrt{:dd + cc - 2cx:} + \sqrt{:dd - xx:}$ will become $b - \sqrt{:dd + cc - 2cx \mp 2ca:} + \sqrt{:dd - xx \mp 2xa - aa:}$; But, since the former is a *Maximum*, it is,

of Consequence, greater than the latter; that is (by Evolution) $b - \sqrt{:dd + cc - 2cx:} + \sqrt{:dd - xx:} \sqsubset b - \sqrt{:dd + cc - 2cx:} - \frac{1}{2} \times \overline{dd + cc - 2cx}|^{\frac{1}{2}-1} \times \mp 2ca - \frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \times \overline{dd + cc - 2cx}|^{\frac{1}{2}-2} \times \overline{\mp 2ca}|^{2}\, o + \sqrt{:dd - xx:} + \frac{1}{2} \times \overline{dd - xx}|^{\frac{1}{2}-1} \times \overline{\mp 2xa - aa} + \frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \times \overline{dd - xx}|^{\frac{1}{2}-2} \times \overline{\mp 2xa - aa}|^{2}\, o$. Which last Step, when reduc'd, gives $\frac{\pm x + \frac{1}{2}a}{\sqrt{:dd - xx:}} + \frac{xxa}{2 \times \overline{dd - xx}|^{\frac{3}{2}}} - \frac{cca}{2 \times \overline{dd + cc - 2cx}|^{\frac{3}{2}}}\, o \sqsubset \frac{\pm c}{\sqrt{:dd + cc - 2cx:}}$: That is to say, $\frac{x}{\sqrt{:dd - xx:}}\, o \sqsubset \frac{c}{\sqrt{:dd + cc - 2cx:}}$

and $\frac{-x}{\sqrt{:dd - xx:}}\, o \sqsubset \frac{-c}{\sqrt{:dd + cc - 2cx:}}$ $\therefore$ $\frac{x}{\sqrt{:dd - xx:}}\, o \sqsupset \frac{c}{\sqrt{:dd + cc - 2cx:}}$; Consequently $\frac{x}{\sqrt{:dd - xx:}}$ is (by our *Lemma* 1.) $= \frac{c}{\sqrt{:dd + cc - 2cx:}}$; Wherefore $ccdd - ccxx = ddxx + ccxx - 2cx^3$ $\therefore$ $2cx^3 - 2c^2xx - ddxx + ccdd = 0$; which Equation being divided by $x - c$ ($= 0$) gives $2cxx - ddx - cdd = 0$: Consequently $x = \frac{dd}{4c} + \sqrt{: \frac{d^4}{16cc} + \frac{dd}{2}:}$

C H A P.

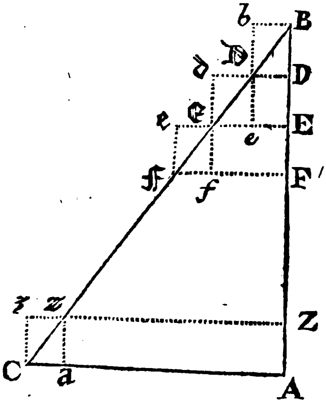
CHAP. III.

Of Quadratures.

PROP. I.

Given the Base A C, and Perpendicular A B of the ∟d △ A B C; required to find its Area.

$AC = b$ } Given. Area of
$BA = p$ }
∟d △ A B C $= \mathfrak{A}$, Required.



Preparation.

Suppose the ⊥ A B to be divided into an indefinite Number of equal Parts B D, D E, E F, &c. and Z A; and suppose each of those Parts $= a$; then thro' the Points B, D, E, F, &c. and Z, draw the Lines B b, D $\mathfrak{D}$ $\mathfrak{d}$, E e $\mathfrak{E}$ $\mathfrak{e}$, F f $\mathfrak{F}$, &c. and Z z $\mathfrak{z}$ ‖s A C; and thro' the Points, wherein the said Lines intersect the Line B C, *viz.* $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, &c. z and C, draw the Lines b $\mathfrak{D}$ e, $\mathfrak{d}$ $\mathfrak{E}$ f, $\mathfrak{e}$ $\mathfrak{F}$, &c. z a, and $\mathfrak{z}$ C ‖s B A: Then the Sum of the ▭s B b $\mathfrak{D}$ D, D $\mathfrak{d}$ $\mathfrak{E}$ E, E $\mathfrak{e}$ $\mathfrak{F}$ F, &c. and Z $\mathfrak{z}$ C A is called the Sum of the * circumscribing ▭s, * because their Sum circumscribes the △; and the Sum of the ▭s D $\mathfrak{D}$ e E, E $\mathfrak{E}$ f F, &c. and Z z a A, is called the Sum of the inscribed ▭s.

Now the first (or least), second, third, &c. of the circumscribing ▭s, being equal to the first, second, third, &c. of the inscribed ▭s respectively, and the Number of the circumscribing ▭ s being (as manifestly it is) $=$ Number of equal Parts, into which A B is, by Supposition, divided $= \frac{AB}{a}$ $\left(= \frac{p}{a}\right)$, and by one more than the Number of the inscribed ▭s; the Sum therefore of the Areas of the circumscribing ▭s is, by the Area of the greatest of them, more than the Sum of the Areas of the inscribed ▭ s.

Solution.

Note, *I suppose it known that the Area of any ▭ is had by multiplying the Length by its perpendicular Height.*

The $\triangle$s BAC, BD$\mathfrak{D}$, BE$\mathfrak{E}$, BF$\mathfrak{F}$, &c. are similar; therefore, by 4. 6. *Eucl. Elements*,

$$p\,(BA) \cdot\cdot\, b\,(AC) :: \begin{cases} a\,(BD) \cdot\cdot\, \dfrac{ba}{p} = D\mathfrak{D} = B\mathfrak{b} \\ 2a\,(BE) \cdot\cdot\, \dfrac{2ba}{p} = E\mathfrak{E} = D\mathfrak{d} \\ 3a\,(BF) \cdot\cdot\, \dfrac{3ba}{p} = F\mathfrak{F} = E\mathfrak{e}, \\ \&c. \end{cases}$$

Wherefore $\frac{ba}{p} \times a \times : 1 + 2 + 3 + 4 +$ &c. continued to $\frac{p}{a}$ Terms $:= \frac{ba}{p} \times a \times : \frac{p}{a} + 1 : \times \frac{p}{2a}$ (by our *Lemma* 2. or by Part 13 or 14.) $= \frac{bp + ba}{2}$ is $=$ the Sum of the Areas of the circumscribing ▭s, and therefore $\sqsubset \mathfrak{A}$: But $\frac{bp+ba}{2} - ba = \frac{bp - ba}{2}$ is $=$ Sum of the Areas of the inscribed ▭s, and therefore $\sqsupset \mathfrak{A}$; wherefore, by our *Lemma* 1. $\frac{bp}{2}$ is $= \mathfrak{A}$.

Scholia.

1. Hence the Area of any $\triangle$ BCD may be had thus, *viz.*

Fig 1

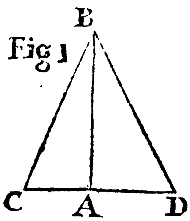
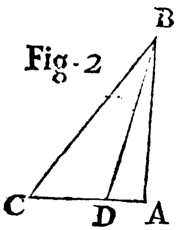


Fig. 2



The Area of the ∟d $\triangle$ BCA is $= \frac{CA \times BA}{2}$ and of the ∟d $\triangle$ BDA is $= \frac{DA \times BA}{2}$, by the foregoing *Proposition*; wherefore the Area of the $\triangle$BCD is $= \frac{CA \times BA}{2} \pm \left\{ \textit{Fig.}\ \begin{matrix}1\\2\end{matrix} \right\} \frac{DA \times BA}{2} = \frac{CD \times BA}{2}$.

2. 'Tis

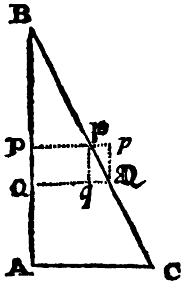
2. 'Tis plain, that if ▭s of equal Baſes be inſcribed in, or circumſcribed about any plane Figure, terminating at one End in a Point, and thence towards the other End continually increaſing, that the Number of the circumſcribing ▭s is = the Number of equal Parts into which the proper Line is ſuppos'd to be divided, and by one more than the Number of the inſcribed ▭s.

3. But, if ▭s of equal Baſes be ſuppos'd to be inſcrib'd in and circumſcribed about any Figure, neither of whoſe Ends terminates in a Point (as C A F *ff*, ſee Fig. to *Prop.* 1.) or if △s, or ſtreight Lines be inſcribed in, and circumſcribed about any Figure whatſoever; the Number of the circumſcribing ▭s, △s or ſtreight Lines is equal to the Number of equal Parts, into which the proper Line is ſuppos'd to be divided, as alſo equal to the Number of the inſcribed ▭s, △s or |s.

4. [Obſerve here,] But if, inſtead of circumſcribing and inſcribing the propos'd Figure with indefinitely little Figures in the foregoing Manner, you only ſuppoſe the proper Line, ſuppoſe A B to be divided into an indefinite Number of equal Parts, each Part being $= a$, and thro' any two of thoſe Diviſions next to one another, at the Diſtances za and $:z-1:\times a$ from (what you make to be) the Beginning of the Line A B, you draw two Lines in ſuch Manner as the Nature of the *Propoſition* requires, and then compleat one of the circumſcribing, along with its inſcribed indefinitely little Figure: Then, having drawn other Lines, ſuch as are (if any be) needful, you will, by the Property of the propos'd Figure, diſcover the Values (*i.e.* the Lengths, Areas or Solidities) of the circumſcribing and inſcribed indefinitely little Figures ſeverally; then by placing 1, 2, 3, 4, &c. to $\frac{AB}{a}$ Terms continued inſtead of z, and univerſally 1^n, 2^n, 3^n, 4^n, &c. inſtead of z^n in each of the ſaid Values, you will have the Values of all the circumſcribing, and inſcribed indefinitely little Figures ſeverally in two, or more Series, whoſe Sums will be had exactly, or *proxime* by our *Lemma* 2. (or by Part 13, or 14.); whence, by our *Lemma* 1. the Value of the propos'd Figure will be found.

By

By the Help of this *Scholium* 4. *Prop.* 1. may be solv'd thus.



$\left.\begin{array}{l} AC = b \\ BA = p \end{array}\right\}$ Given.

Area of $\llcorner$d $\triangle$ ABC $= \mathfrak{A}$, required.

Suppose the $\perp$ BA to be divided into an indefinite Number of equal Parts, each Part being $= a$ ($=$ PQ); then at the indetermined Distances BQ $= za$ and BP $=$ $:z-1:\times a$ from B, draw the Lines $Qq\mathfrak{Q}$ and $P\mathfrak{P}p$ $\perp$s BA, and intersecting BC in $\mathfrak{Q}$ and $\mathfrak{P}$, and draw the Lines $\mathfrak{P}q$, $p\mathfrak{Q}$ $\parallel$s BA. Then the $\triangle$s BAC, $BQ\mathfrak{Q}$, and $BP\mathfrak{P}$ are similar; wherefore,

$$p\,(AB) \cdot\cdot\, b\,(AC) :: \left\{\begin{array}{l} za\,(BQ) \cdot\cdot \dfrac{bza}{p} = Q\mathfrak{Q} = Pp. \\ :z-1:\times a\,(BP) \cdot\cdot \dfrac{bza-ba}{p} = P\mathfrak{P} = Qq \end{array}\right.$$

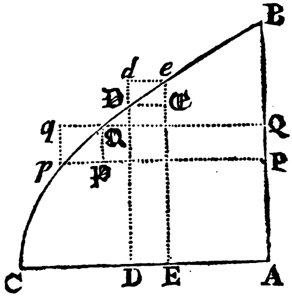
Whence $\frac{bza}{p} \times a$ is $=$ Area of the $\square$ $Pp\mathfrak{Q}Q$, and $\frac{bza-ba}{p} \times a$ is $=$ Area of $\square$ $P\mathfrak{P}qQ$; wherefore, by *Scholium* 4. $\frac{baa}{p} \times : 1 + 2 + 3 + 4 +$ &c. continued to $\frac{p}{a}$ Terms: $= \frac{bp+ba}{2}$ (by our *Lemma* 2. or * by Part 13 or 14.) is $=$ Sum of the Areas of the circumscribing $\square$s, and therefore $\sqsubset \mathfrak{A}$: And $\frac{baa}{p} \times : 0 + 1 + 2 + 3 +$ &c. continued to $\frac{p}{a}$ Terms: $= \frac{bp-ba}{2}$ (by our *Lem.* 2, or by Part 13 or 14.) $=$ the Sum of the Areas of the inscribed $\square$s, and therefore $\sqsupset \mathfrak{A}$; consequently, by our *Lemma* 1. $\frac{pb}{2} = \mathfrak{A}$.

* Book I.

PROP.

PROP. II.

Given the Abscisse $CA = x$ of the parabolick Figure CAB, and consequently, from the given Property of the Figure, the Ordinate $AB = y$. Required to find the Area of the said Figure.



Let the Property of the Figure ABC be such, that drawing any ordinate $D\mathfrak{D}$ ($\perp CA$ and $\parallel AB$), the mth Power thereof may be equal to its respective Abscisse, *viz.* $\overline{D\mathfrak{D}}|^m = CD$, m being $=$ any given affirmative whole Quantity.

Suppose the Ordinate AB to be divided into an indefinite Number of equal Parts, each Part being $= a$ ($= PQ$), and thro' any two, next to one another, of those Divisions, as Q and P, at the Distances $za = AQ$ and $:z - 1: \times a = AP$ from A, draw the Lines $Q\mathfrak{Q}q$ and $P\mathfrak{P}p$ $\perp$s AB, and intersecting the Curve in $\mathfrak{Q}$ and p; and draw the Lines qp and $\mathfrak{Q}\mathfrak{P}$ $\parallel$s BA.

Solution.

Then, 'tis manifest, by the Property of the Figure, that $AC - Q\mathfrak{Q} = \overline{AQ}|^m$; consequently $x - \overline{za}|^m$ ($= AC - \overline{AQ}|^m$) $= \mathfrak{Q}Q$; and $x - \overline{:z-1:\times a}|^m$ ($= AC - \overline{AP}|^m$) $= pP$. Whence $a \times : x - \overline{za}|^m$: is $=$ Area of the inscribed $\square$ $P\mathfrak{P}\mathfrak{Q}Q$; and $a \times : x - \overline{\overline{z-1} \times a}|^m$: is $=$ Area of the $\square$ $QqpP$: Wherefore, by *Schol.* 4. $a \times : \overline{x - a^m} + \overline{x - \overline{2a}|^m} + \overline{x - \overline{3a}|^m} + \overline{x - \overline{4a}|^m} +$ &c. to $\frac{y}{a}$ Places continued: (the two Terms under each Line of Connexion being what is here call'd a Place) $= a \times x \times \frac{y}{a} - a^{m+1} \times \frac{\overline{\frac{y}{a}}|^{m+1}}{m+1} - \frac{1}{2} \times \overline{\frac{y}{a}}|^m \phi$, by Part 13 or 14, or by 1st *Corollary* of our *Lemma* 2, $= xy - \frac{y^{m+1}}{m+1} - \frac{y^m a}{2}\phi = \frac{myx}{m+1} - \frac{1}{2} x a \phi$, since

since y^m is equal to x, is $=$ Sum of the Areas of the inscribed $\Box$s, which is therefore $\sqsupset$ Area of the Figure A B C.

And $a \times : \overline{x - 0} + \overline{x - a}|^m + \overline{x - 2a}|^m + \overline{x - 3a}|^m$ + &c. to $\frac{y}{a}$ Places continued : is manifestly $=$ Sum of the inscribed $\Box$s $+ a^{m+1} \times \overline{\frac{y}{a}}|^m = \frac{myx}{m+1} + \frac{1}{2} x a \phi$ (since y^m is equal to x) is $=$ the Sum of the Areas of the circumscribing $\Box$s, which is therefore $\sqsubset$ Area of the Figure A B C; consequently (by our *Lemma* 1.) $\frac{m}{m+1} y x$ is $=$ Area of the Figure A B C.

Scholium.

If the Abscisse C A had been suppos'd to be divided into any indefinite Number of equal Parts, each Part being $= a$, and thro' any two Points D and E of those Divisions, at the Distances : $z - 1 : \times a$ and $z a$ from C, the Lines D 𝔇 d and E 𝔈 e were drawn $\perp$s C A; and the Lines 𝔇 𝔈, $d e \parallel$s C A; then the Area of the $\Box$ D E e d being $= \overline{za}|^{\frac{1}{m}} \times a$; and the Area of the $\Box$ D E 𝔈 𝔇 being $= : \overline{z - 1 : \times a}|^{\frac{1}{m}} \times a$; the Sum of the Areas of the circumscribing $\Box$s $= a^{\frac{1}{m}} \times a \times : 1^{\frac{1}{m}} + 2^{\frac{1}{m}} + 3^{\frac{1}{m}} + 4^{\frac{1}{m}}$ + &c. continued to $\frac{x}{a}$ Terms : must be $= \frac{mxy}{m+1} \circ = \frac{m}{m+1} x^{\frac{m+1}{m}} \circ = \frac{x^{\frac{1}{m}+1}}{\frac{1}{m}+1} \circ =$

$$a^{\frac{1}{m}+1} \times \frac{\overline{\frac{x}{a}}|^{\frac{1}{m}+1}}{\frac{1}{m}+1} \circ$$

(by the preceding *Prop.*): And the Sum of the Areas of the inscribed $\Box$s $= a^{\frac{1}{m}} \times a \times : 0 +$

$1^{\frac{1}{m}}$

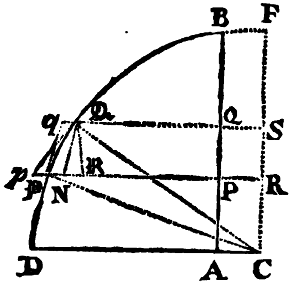
$1^{\frac{1}{m}} + 2^{\frac{1}{m}} + 3^{\frac{1}{m}} +$ &c. continued to $\frac{x}{a}$ Terms: must be $=$

$a^{\frac{1}{m}+1} \times \dfrac{\overline{\frac{x}{a}}\big|^{\frac{1}{m}+1}}{\frac{1}{m}+1}$ $\ominus$: Whence it is manifest that $1^{\frac{1}{p}} + 2^{\frac{1}{p}}$ $+ 3^{\frac{1}{p}} +$ &c. continued to $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}$ Terms (n being indefinite, and $\frac{1}{p}$ equal to any affirmative Fraction, whose Numerator is 1) is $= \dfrac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} \left\{\begin{matrix} \ominus \\ \ominus \end{matrix}\right\}$

Scholium II.

The Sum of 1^p, 2^p, 3^p, 4^p, &c. continued to $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}$ Terms (n being equal to an indefinite Number) being $=$ $\dfrac{n^{p+1}}{p+1} \left\{\begin{matrix} \ominus \\ \ominus \end{matrix}\right\}$ when p is equal to any affirmative integer Number (by Part 13 or 14); as also that Canon being $=$ Sum of that Series, when p is equal to any affirmative Fraction whose Numerator is 1 (by the above *Schol.* 1.); it is manifest that the said Canon is $=$ the Sum of the said Series, when p is equal to any intermediate Number between the said integer and fracted Value thereof: But this intermediate Number is capable of being equal to any other numerical Fraction whatever, which demonstrates, without any more ado, that the above Canon expresses the Sum of the above Series, when p is equal to any affirmative Number whatsoever. [See our *Lemma* 2.]

PROP. III.



Given the Sine AB of the Arc BD of a Circle, as alſo the Radius DC: Required to find the Length of the Arc DB.

$\left.\begin{array}{l} BA = s \\ DC = r \end{array}\right\}$ Given.

$DB = \mathfrak{a}$. Required.

Solution I.

Suppoſe the Line (or Sine) AB to be divided into an indefinite Number of equal Parts, and each Part ($= PQ$) $= a$; thro' any two Points Q and P of which Diviſions, at the Diſtances $za = AQ$ and $:z - 1: \times a = AP$ from A, draw the Lines $SQ\mathfrak{Q}q$, and $RP\mathfrak{R}N\mathfrak{P}p$ ⊥s AB: Draw the Line $\mathfrak{Q}p$ ſo as to touch the Arc in $\mathfrak{Q}$, and interſect Rp in p; Draw alſo the Line $\mathfrak{P}q$ ſo as to touch the Arc in $\mathfrak{P}$; and draw $N\mathfrak{Q} \parallel \mathfrak{P}q$: Draw the Lines CRSF and $\mathfrak{R}\mathfrak{Q} \parallel$s AB; and laſtly draw the Lines $\mathfrak{Q}C$ and $\mathfrak{P}C$. Then 'tis plain that the indefinitely ſhort circumſcribing Line $\mathfrak{Q}p$ is greater than the Tangent of its reſpective Arc, and conſequently greater or longer than $\mathfrak{Q}\mathfrak{P}$; and that their reſpective indefinitely ſhort inſcribed Line $\mathfrak{Q}N$ is leſs than the Sine of the Arc $\mathfrak{Q}\mathfrak{P}$, and conſequently leſs or ſhorter than $\mathfrak{Q}\mathfrak{P}$.

The ∟d △s $C\mathfrak{Q}S$ and $p\mathfrak{Q}\mathfrak{R}$ are ſimilar; for the $\angle p\mathfrak{Q}\mathfrak{R}$ = Complement of $\angle C\mathfrak{Q}\mathfrak{R}$ to a ∟ is $= C\mathfrak{Q}S$; conſequently $\sqrt{:rr - z^2a^2:}\ (\mathfrak{Q}S) \cdot\cdot\ r\ (C\mathfrak{Q}) :: a\ (\mathfrak{Q}\mathfrak{R})$

$$\cdot\cdot\ \frac{ra}{\sqrt{:rr - z^2a^2:}} = p\mathfrak{Q}.$$

Again, the ∟d △s $N\mathfrak{Q}\mathfrak{R}$ and $R\mathfrak{P}C$ are ſimilar; for the $\angle \mathfrak{Q}N\mathfrak{R} = \angle q\mathfrak{P}\mathfrak{R}$ = Complement of $\angle R\mathfrak{P}C$ to a ∟ is $= \angle \mathfrak{P}CR$; wherefore $\sqrt{:rr - \overline{z-1}|^2 a^2:}\ (\mathfrak{P}R)$

$$\cdot\cdot\ r\ (\mathfrak{P}C) :: a\ (\mathfrak{Q}\mathfrak{R})\ \cdot\cdot\ \frac{ra}{\sqrt{:rr - \overline{z-1}|^2 a^2:}} = \mathfrak{Q}N;$$

wherefore, (by *Scholium* 4) $ra \times :\overline{rr - aa}|^{-\frac{1}{2}} + \overline{rr - 2^2a^2}|^{-\frac{1}{2}}$

$+ \overline{rr - 3^2a^2}|^{-\frac{1}{2}} +$ &c. to $\frac{s}{a}$ Places continued: is = the Sum of the circumſcribing Lines, and therefore $\sqsubset \mathfrak{a}$: And

$ra \times$

$ra \times : \overline{rr-0}|^{-\frac{1}{2}} + \overline{rr-a^2}|^{-\frac{1}{2}} + \overline{rr-2^2 a^2}|^{-\frac{1}{2}} +$ &c. to $\frac{s}{a}$ Places continued: is $=$ the Sum of the inscribed Lines, and is therefore $\sqsupset \mathfrak{A}$. Whence the Sum of the circumscribing Lines exceeds that of the inscribed Lines only by the indefinitely small Quantity $ra \times : \overline{rr-ss}|^{-\frac{1}{2}} - \overline{rr-0}|^{-\frac{1}{2}}:$; Wherefore the Sum of either Series is indefinitely near $= \mathfrak{A}$.

But the Sum of either Series cannot be had in infinite Terms; Wherefore we must have recourse to Approximations by an infinite Series; thus $\mathfrak{Q}p = ra \times \overline{rr - z^2 a^2}|^{-\frac{1}{2}}$ is (by Sir *Is.* *Newton*'s Theorem) $= ra \times : \overline{rr}|^{-\frac{1}{2}} \left(\frac{1}{r}\right) + \frac{z^2 a^2}{2r^3} + \frac{3z^4 a^4}{8r^5} + \frac{5z^6 a^6}{16r^7} +$ &c. continued *in infinitum*:; wherefore (by *Scholium* 4.)

$$ra \times : \frac{1}{r} + \frac{a^2}{2r^3} + \frac{3a^4}{8r^5} + \frac{5a^6}{16r^7} + \text{\&c. } \textit{in infinit.}:$$

$$+ ra \times : \frac{1}{r} + \frac{2^2 a^2}{2r^3} + \frac{3 \times 2^4 a^4}{8r^5} + \frac{5 \times 2^6 a^6}{16r^7} + \text{\&c. } \textit{in infinit.}:$$

$$+ ra \times : \frac{1}{r} + \frac{3^2 a^2}{2r^3} + \frac{3 \times 3^4 a^4}{8r^5} + \frac{5 \times 3^6 a^6}{16r^7} + \text{\&c. } \textit{in infinit.}:$$

$+$ &c. to $\frac{s}{a}$ Series continued

$= s + \frac{s^3}{6r^2} + \frac{s^2 a}{4r^2}\phi + \frac{3s^5}{40r^4} + \frac{3s^4 a}{16r^4}\phi + \frac{5s^7}{112r^6} + \frac{5s^6 a}{32r^6}\phi +$ &c. *in infinitum* (by adding the first Members of each Series into one Sum, also the second Members into another, also the third Members into another, &c. by our *Lemma* 2.) is $=$ the Sum of the circumscribing Lines, and therefore $\sqsubset \mathfrak{A}$. And the Sum of the circumscribing Lines $- ra \times : \overline{rr-ss}|^{-\frac{1}{2}} - \overline{rr}|^{-\frac{1}{2}} := s + \frac{s^3}{6r^2} - \frac{s^2 a}{4rr}\phi + \frac{3s^5}{40r^4} - \frac{3s^4 a}{16r^4}\phi + \frac{5s^7}{112r^6} - \frac{5s^6 a}{32r^6}\phi$, &c. *in infinitum* is (by what has been before said) $=$ Sum of the inscribed Lines, and there-

therefore $\sqsupset \mathfrak{a}$; Consequently (by our *Lemma* 1.) $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} + \frac{5s^7}{112r^6} + \frac{35s^9}{1152r^8} +$ &c. *sine fine* is $= \mathfrak{a}$.

That is to say, If you reduce $\overline{rr - ss}|^{-\frac{1}{2}}$ to an infinite Series, and multiply each Member of that Series by sr, and then divide each Member of the Product by the Index of s in that Member, you will have the Value of $\mathfrak{a}$ required.

Or putting A, B, C, D, &c. for the first, second, third, fourth and following Terms

$$\mathfrak{a} \text{ is} = s + \frac{1 \times 1 \times ss}{2 \times 3rr} A + \frac{3 \times 3ss}{4 \times 5rr} B + \frac{5 \times 5\,ss}{6 \times 7rr} C + \frac{7 \times 7\,ss}{8 \times 9rr} D + \text{&c.}$$

Scholium I.

$a = \mathfrak{a}$ being given and s sought; 'tis required to find s by the Help of the above Series.

If s be suppos'd $= b\,a + c\,a^2 + d\,a^3 + e\,a^4 +$ &c; then b will (by the following Method) be found $= 1$, and $c = 0 = e = g = i$, &c; and therefore suppose

$$s = a + d a^3 + f a^5 + h a^7 + k a^9, \text{&c.}$$

$$\left.\begin{array}{l} \text{then } \frac{s^3}{6rr} = \frac{a^3}{6rr} + \frac{d}{2rr}a^5 + \frac{f + dd}{2rr}a^7 + \frac{3h + 6df + d^3}{6rr}a^9, \text{&c.} \\ \frac{3s^5}{40r^4} = \frac{3a^5}{40r^4} + \frac{3d}{8r^4}a^7 + \frac{3f + 6dd}{8r^4}a^9, \text{&c.} \\ \frac{5s^7}{112r^6} = \frac{5a^7}{112r^6} + \frac{35d}{112r^6}a^9, \text{&c.} \\ \frac{35s^9}{1152r^8} = \frac{35a^9}{1152r^8}, \text{&c.} \end{array}\right\} = a$$

&c.

Now by equating the respective Terms, we have

$$a = a.$$

Also $d a^3 + \frac{a^3}{6rr} = 0$; consequently $d = -\frac{1}{6rr}$.

In like Manner f will be found $= \frac{1}{120r^4}$.

Also

Alſo $h = -\frac{1}{5040 r^6}$

Likewiſe $k = \frac{1}{362880 r^8}$
&c.

Conſequently $s = a - \frac{a^3}{6 r r} + \frac{a^5}{120 r^4} - \frac{a^7}{5040 r^6} + \frac{a^9}{362880 r^8}$, &c. $= a - \frac{a a}{2 \times 3 r r} A - \frac{a a}{4 \times 5 r r} B - \frac{a a}{6 \times 7 r r} C - \frac{a a}{8 \times 9 r r} D -$ &c. putting A, B, C, D, &c. for the firſt, ſecond, third, fourth, &c. Terms.

* By this Series the natural Sine of any Arc to any aſſumed Radius may be computed immediatly from having only the Length of the Arc given.

Scholium II.

* Likewiſe the Σ, or Co-ſine being $= \sqrt{:rr - ss:}$, it is therefore $= \sqrt{: r r - a^2 + \frac{a^4}{3 r r} - \frac{2 a^6}{45 r^4} + \frac{a^8}{315 r^6} - \frac{2 a^{10}}{14175 r^8}}$, &c: $= r - \frac{a^2}{2 r} + \frac{a^4}{24 r^3} - \frac{a^6}{720 r^5} + \frac{a^8}{40320 r^7} - \frac{a^{10}}{3628800 r^9}$, &c. $= r - \frac{a^2}{1 \times 2 r r} A - \frac{a^2}{3 \times 4 r r} B - \frac{a^2}{5 \times 6 r r} C - \frac{a^2}{7 \times 8 r r} D - \frac{a^2}{9 \times 10 r r} E -$ &c. putting A, B, C, D, E, &c. for the firſt, ſecond, third, fourth, fifth, &c. Terms.

Scholium III.

$\Sigma \cdot\cdot S :: R \cdot\cdot T$; conſequently, Radius being $= 1$, the Tangent is = the Series expreſſing its Value in Page 324.
&c.

Scholium IV.

The Sine of an Arc being given; 'tis required to find the Sine of another Arc that ſhall be to the firſt, as n to 1.

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* *Vide Pages* 323, 324.

Let

Let s be the Sine given, and z the Sine required; then the Arc belonging to the Sine s, is $s+\frac{s^3}{6rr}+\frac{3s^5}{40r^4}+\frac{5s^7}{112r^6}$ + &c. And the Arc belonging to the Sine z, is $z+\frac{z^3}{6rr}+\frac{3z^5}{40r^4}+\frac{5z^7}{112r^6}$ + &c. And the first of these Arcs is to the second, as 1 to n; consequently $ns+\frac{ns^3}{6rr}+\frac{3ns^5}{40r^4}+\frac{5ns^7}{112r^6}+$ &c. $=z+\frac{z^3}{6rr}+\frac{3z^5}{40r^4}+\frac{5z^7}{112r^6}+$ &c.

Now suppose $z=bs+ds^3+fs^5+hs^7+$ &c. then

$$\left.\begin{array}{lllll} z= bs+ds^3 & +fs^5 & +hs^7 & + \&c. \\ \frac{z^3}{6rr}= \frac{b^3}{6rr}s^3 & +\frac{bbd}{2rr}s^5 & +\frac{bd^2+b^2f}{2rr}s^7 & + \&c. \\ \frac{3z^5}{40r^4}= & \frac{3b^5}{40r^4}s^5 & +\frac{3b^4d}{8r^4}s^7 & + \&c. \\ \frac{5z^7}{112r^6}= & & \frac{5b^7}{112r^6}s^7 & + \&c. \\ & & \&c. & \end{array}\right\} = ns+\frac{n}{6rr}s^3+\frac{3n}{40r^4}s^5+\frac{5n}{112r^6}s^7 + \&c.$$

Now by equating the respective Terms with each other, we have

$$bs=ns;\ \text{consequently}\ b=n$$

Also $ds^3+\frac{b^3}{6rr}s^3=\frac{n}{6rr}s^3 \because d=\frac{n-n^3}{6rr}$

In like Manner f is found $=\frac{9n-10n^3+n^5}{120r^4}$

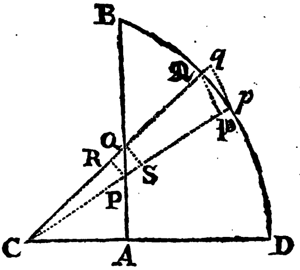
Also $h=\frac{225n-259n^3+35n^5-n^7}{5040r^6}$

&c.

Conseq. $z=ns+\frac{n-nnn}{6rr}s^3+\frac{9n-10n^3+n^5}{120r^4}s^5+\frac{225n-259n^3+35n^5-n^7}{5040r^6}s^7+$ &c. $=ns+\frac{1^2-n^2}{2\times 3rr}s^2A+\frac{3^2-n^2}{4\times 5rr}s^2B+\frac{5^2-n^2}{6\times 7rr}s^2C+$ &c. putting A, B, C, &c. for the first, second, third, &c. Terms; where it is observable

ble that if n be an odd Number, the Series will be finite. Vide *Probl.* 9. *Chap.* 3. *Part* II.

Solution II.



Suppose the Line (or Sine) AB to be divided into an indefinite Number of equal Parts, and each Part $=a$ ($=$ PQ): and thro' any two Points Q and P of those Divisions, at the Distances $za =$ AQ and : $z-1$: $\times a =$ AP from A, draw the Lines CQ𝔔q and CP𝔓p, intersecting AB in Q and P, and the Arc BD in 𝔔 and p: Draw the Lines qp, 𝔔𝔓 and QS $\perp$s Cp, and PR $\perp$ Cq: And suppose AC ($=\sqrt{:rr-ss:}$) $=c$.

The $\triangle$s CAQ and PQR are similar; wherefore $\sqrt{:cc+z^2a^2:}$ (CQ) $\cdot\cdot$ c (AC) :: a (PQ) $\cdot\cdot$ $\dfrac{ca}{\sqrt{:cc+z^2a^2:}} =$ PR.

And the $\triangle$s CPR and C𝔔𝔓 are similar; wherefore $\sqrt{:cc+z^2a^2:}$ (CQ) $\cdot\cdot$ $\dfrac{ca}{\sqrt{:cc+z^2a^2:}}$ (PR) :: r (C𝔔) $\cdot\cdot$ $\dfrac{rca}{cc+zzaa}$ which (because CQ is greater than CP) is $\sqsupset$ inscribed Line (or Sine) 𝔔𝔓.

Again, the $\triangle$s CPA and QPS are similar; wherefore $\sqrt{:cc+\overline{z-1}|^2a^2:}$ (CP) $\cdot\cdot$ c (AC) :: a (PQ) $\cdot\cdot$ $\dfrac{ca}{\sqrt{:cc+\overline{z-1}|^2a^2}} =$ SQ.

And the $\triangle$s CSQ and Cpq are similar; wherefore $\sqrt{:cc+\overline{z-1}|^2a^2:}$ (CP) $\cdot\cdot$ $\dfrac{ca}{\sqrt{:cc+\overline{z-1}|^2a^2}}$ (SQ) :: r (Cp) $\cdot\cdot$ $\dfrac{rca}{cc+\overline{z-1}|^2a^2}$ which (because CS is greater than CP) is $\sqsubset$ the circumscribing Line (or Tangent) pq; Wherefore (by *Schol.* 4.) $rca\times:\dfrac{1}{cc+aa}+\dfrac{1}{c^2+2^2a^2}+\dfrac{1}{cc+3^2a^2}+$&c.

to $\frac{s}{a}$ Places continued: is $\sqsupset$ Sum of the inscribed Lines, and consequently $\sqsupset \mathfrak{A}$: And $rca \times : \frac{1}{cc} + \frac{1}{cc+aa} + \frac{1}{cc+2^2a^2}$ + &c. continued to $\frac{s}{a}$ Places: is $\sqsubset$ Sum of the circumscribing Lines, and consequently $\sqsubset \mathfrak{A}$.

Hence 'tis manifest that the Sum of the last (or greater) Series exceeds that of the first (or lesser) Series only by the indefinitely small Quantity $rca \times : \frac{1}{cc} - \frac{1}{cc+ss} :$; Wherefore the Sum of either Series is indefinitely near $= \mathfrak{A}$: But the Sum of either Series cannot be had in finite Terms; wherefore we must have recourse to Approximations; thus $\frac{rca}{cc+zzaa}$ is (by Division) $= rca \times : \frac{1}{cc} - \frac{z^2a^2}{c^4} + \frac{z^4a^4}{c^6} - \frac{z^6a^6}{c^8}$; &c. *in infinitum* :; Wherefore (by *Soholium* 4.)

$$rca \times : \frac{1}{cc} - \frac{a^2}{c^4} + \frac{a^4}{c^6} - \frac{a^6}{c^8}, \text{ \&c. } in\ infinitum :$$

$$+ rca \times : \frac{1}{cc} - \frac{2^2a^2}{c^4} + \frac{2^4a^4}{c^6} - \frac{2^6a^6}{c^8}, \text{ \&c. } in\ infinitum :$$

$$+ rca \times : \frac{1}{cc} - \frac{3^2a^2}{c^4} + \frac{3^4a^4}{c^6} - \frac{3^6a^6}{c^8}, \text{ \&c. } in\ infinitum :$$

+ &c. to $\frac{s}{a}$ Series continued

$$= \frac{rs}{c} - \frac{rs^3}{3c^3} - \frac{rs^2a}{2c^3}\phi + \frac{rs^5}{5c^5} + \frac{rs^4a}{2c^5}\phi - \frac{rs^7}{7c^7} - \frac{rs^6a}{2c^7}\phi,$$

&c. *in infinitum* (by our *Lemma* 2.) is $\sqsupset$ Sum of the inscribed Lines (or Sines) and consequently $\sqsupset \mathfrak{A}$.

But the Sum of the foregoing Series, and of the indefinitely little Quantity $rca \times : \frac{1}{cc} - \frac{1}{cc+ss} : =$

$$\frac{rs}{c} - \frac{rs^3}{3c^3} + \frac{rs^2a}{2c^3}\phi + \frac{rs^5}{5c^5} - \frac{rs^4a}{2c^5}\phi - \frac{rs^7}{7c^7} + \frac{rs^6a}{2c^7}\phi, \text{\&c.}$$

is $\sqsubset$ Sum of the circumscribing Lines (or Tangents), and therefore $\sqsubset \mathfrak{A}$.

Now,

Now, since the latter Sum is $\sqsubset$ former; $\frac{rs^2a}{2c^3} - \frac{rs^4a}{2c^5} + \frac{rs^6a}{2c^7}$, &c. *in infinitum* is affirmative (but withal indefinitely little); for it is $\sqsubset - \frac{rs^2a}{2c^3} + \frac{rs^4a}{2c^5} - \frac{rs^6a}{2c^7}$ &c. *in infinitum* which is therefore negative: Consequently (by our *Lemma* 1.) $\frac{rs}{c} - \frac{rs^3}{3c^3} + \frac{rs^5}{5c^5} - \frac{rs^7}{7c^7}$; &c. *Sine fine* is $= a$.

That is to say, If you reduce $\frac{rsc}{cc+ss}$ to an infinite Series (which is done by Division), and then divide each Member of that Series by the Index of s in that Member, you will have the Value of a required.

By either of the two foregoing Solutions the Ratio of the Diameter of a Circle to its Circumference may be expeditiously found and express'd near the Truth in Numbers: Thus by the last Solution,

Suppose the Arc BD $= 30^\circ\ 00'$, and r ($=$ Radius) $= 1$; then s will be $= .5$, and $c = \sqrt{.75}$; and $6 \times$ BD will be $= 180^\circ\ 00' = 6 \times : \frac{.5}{\sqrt{.75}} - \frac{.5^3}{3\sqrt{.75}^3} + \frac{.5^5}{5\sqrt{.75}^5}$, &c: $= 6 \times \frac{.5}{\sqrt{.75}} (= \sqrt{12}) \times : 1 - \frac{1}{3 \times 3} + \frac{1^2}{5 \times 3^2} - \frac{1^3}{7 \times 3^3}$, &c.: $= \sqrt{12} - 3 \times 3) \sqrt{12} + 5 \times 3^2) \sqrt{12} - 7 \times 3^3) \sqrt{12} + 9 \times 3^4) \sqrt{12}$, &c; Wherefore $1 \cdot\cdot \sqrt{12} - 3 \times 3) \sqrt{12} + 5 \times 3^2) \sqrt{12}$, &c :: Radius $\cdot\cdot$ 180° :: Diameter $\cdot\cdot$ Circumference of any Circle.

Operation.

	$\sqrt{12} =$	3.46410161513775487054
3	$)\sqrt{12} =$	1.15470053837925152901 8
3^2	$)\sqrt{12} =$	.384900179459750509672
3^3	$)\sqrt{12} =$	.128300059819916836557
3^4	$)\sqrt{12} =$	42766686606638945519
3^5	$)\sqrt{12} =$	14255562202212981839
3^6	$)\sqrt{12} =$	4751854067404327279
3^7	$)\sqrt{12} =$	1583951355801442426
3^8	$)\sqrt{12} =$	527983785267147475
3^9	$)\sqrt{12} =$	175994595089049158
3^{10}	$)\sqrt{12} =$	58664865029683052
3^{11}	$)\sqrt{12} =$	19554955009894350
3^{12}	$)\sqrt{12} =$	6518318336631450
3^{13}	$)\sqrt{12} =$	2172772778877150
3^{14}	$)\sqrt{12} =$	724257592959050
3^{15}	$)\sqrt{12} =$	241419197653016
3^{16}	$)\sqrt{12} =$	80473065884338
3^{17}	$)\sqrt{12} =$	26824355294779
3^{18}	$)\sqrt{12} =$	8941451764926
3^{19}	$)\sqrt{12} =$	2980483921642
3^{20}	$)\sqrt{12} =$	993494640547
3^{21}	$)\sqrt{12} =$	331164880182
3^{22}	$)\sqrt{12} =$	110388293394
3^{23}	$)\sqrt{12} =$	36796097798
3^{24}	$)\sqrt{12} =$	12265365932
3^{25}	$)\sqrt{12} =$	4088455310
3^{26}	$)\sqrt{12} =$	1362818436
3^{27}	$)\sqrt{12} =$	454272812
3^{28}	$)\sqrt{12} =$	151424270
3^{29}	$)\sqrt{12} =$	50474756
3^{30}	$)\sqrt{12} =$	16824918
3^{31}	$)\sqrt{12} =$	5608306
3^{32}	$)\sqrt{12} =$	1869435
3^{33}	$)\sqrt{12} =$	623145
3^{34}	$)\sqrt{12} =$	207715
3^{35}	$)\sqrt{12} =$	69238
3^{36}	$)\sqrt{12} =$	23079
3^{37}	$)\sqrt{12} =$	7693
3^{38}	$)\sqrt{12} =$	2564
3^{39}	$)\sqrt{12} =$	854
3^{40}	$)\sqrt{12} =$	284
3^{41}	$)\sqrt{12} = 3^{40})\sqrt{12} \div 3) =$	94

√12

$\sqrt{12} =$	3.46410161513775487054
$5\times3^{2}\ )\sqrt{12} =$	76980035891950101934
$9\times3^{4}\ )\sqrt{12} =$	4751854067404327279
$13\times3^{6}\ )\sqrt{12} =$	365527235954179021
$17\times3^{8}\ )\sqrt{12} =$	31057869721596910
$21\times3^{10})\sqrt{12} =$	2793565001413478
$25\times3^{12})\sqrt{12} =$	260732733465258
$29\times3^{14})\sqrt{12} =$	24974399757208
$33\times3^{16})\sqrt{12} =$	2438577754070
$37\times3^{18})\sqrt{12} =$	241660858511
$41\times3^{20})\sqrt{12} =$	24231576598
$45\times3^{22})\sqrt{12} =$	2453073186
$49\times3^{24})\sqrt{12} =$	250313590
$53\times3^{26})\sqrt{12} =$	25713555
$57\times3^{28})\sqrt{12} =$	2656566
$61\times3^{30})\sqrt{12} =$	275818
$65\times3^{32})\sqrt{12} =$	28760
$69\times3^{34})\sqrt{12} =$	3010
$73\times3^{36})\sqrt{12} =$	316
$77\times3^{38})\sqrt{12} =$	33
$81\times3^{40})\sqrt{12} =$	3
Sum $=$	3.546233172182121682158
$3\times3\ )\sqrt{12} =$	.38490017945975050967 2
$7\times3^{3}\ )\sqrt{12} =$	1832857999742738333793
$11\times3^{5}\ )\sqrt{12} =$	129596020020118167
$15\times3^{7}\ )\sqrt{12} =$	10559675705342949 5
$19\times3^{9}\ )\sqrt{12} =$	926287342573942 9
$23\times3^{11})\sqrt{12} =$	850215435212797
$27\times3^{13})\sqrt{12} =$	80473065884338
$31\times3^{15})\sqrt{12} =$	7787716053323
$35\times3^{17})\sqrt{12} =$	766410151279
$39\times3^{19})\sqrt{12} =$	76422664657
$43\times3^{21})\sqrt{12} =$	7701508841
$47\times3^{23})\sqrt{12} =$	782895697
$51\times3^{25})\sqrt{12} =$	80165790
$55\times3^{27})\sqrt{12} =$	8259505
$59\times3^{29})\sqrt{12} =$	855504
$63\times3^{31})\sqrt{12} =$	89020
$67\times3^{33})\sqrt{12} =$	9300
$71\times3^{35})\sqrt{12} =$	975
$75\times3^{37})\sqrt{12} =$	102
$79\times3^{39})\sqrt{12} =$	10
$83\times3^{41})\sqrt{12} =$	1
Sum $=$	.40464051859232844369 | 5
Diff. of the two foreg. Sums $=$	3.14159265358979323846 | 3

That is the Diameter of any Circle is to its Circumference, as 1 to 3.14159265358979323846 +.

Scholium.

Having thus found the Ratio of the Diameter of a Circle to its Circumference to be As 1 to 3.14159 &c. it will not be improper to shew how to find the Ratio's nearest to it in the least integer Terms; and then to choose from among them that which is fittest for Practice. In the doing of which it will appear, that the exact Ratio of the Diameter of a Circle to its Circumference can never be express'd in rational Numbers.

Proposition.

Let it be required to find the nearest Ratio in Integer Terms consisting of not above ten Figures of 1 to 3.1415926535898—; *viz.* of the Ratio of the Diameter of a Circle to its Circumference.

The adjoining Ratio's in the least Integers to the given one, are manifestly 1 ·· 4 and 1 ·· 3, which being truly compleated are

1	1 ·· 4 —.85840 &c.
2	1 ·· 3 +.14159 &c.

Now the Fraction in the 2d Step being (secluding their Signs) less than that in the 1st, and the first Term of each being the same, *viz.* 1; the Ratio therefore of the Integers in the 2d, is nearer the Truth than that of the Integers in the 1st: Whence the 2d Step is called the I continual Increment; for its respective Terms are to be continually added to them of the 1st, till a Step be thus had in which the Fraction shall be less than that in the I cont. Inc., which Step, when found, will be the II cont. Inc. And the other Steps found by the said Additions may be called the intermediate Steps: thus,

1+2	3	2 ·· 7 —.71681	&c. intermediate Steps.
3+2	4	3 ·· 10 —.57522	
4+2	5	4 ·· 13 —.43362	
5+2	6	5 ·· 16 —.29203	
6+2	7	6 ·· 19 —.15044	
7+2	8	7 ·· 22 —.00885	I cont. Inc.

Again, adding the respective Terms of the II. cont. Inc. continually to those of the I. till a Step be thus found, whose Fraction shall be less than that in the II cont. Inc., this Step will be the III cont. Inc. &c.

This Method, duly consider'd, carries its own Demonstration with it: But, on the other Hand, it is very tedious, as also superfluous: For the Ratio of the integer Parts in any cont. Inc. is nearer the Truth than the Ratio exhibited by the integer Parts in any of the one half almost of the next succeeding intermediate Steps; and in the cont. Inc. the Fractions are least at which Junctures the Ratios of the integer Parts must be best: the better Way therefore is to deduce from the said Method a Rule for finding all the cont. Inc. exclusive of the intermediate Steps; and then &c.

RULE.

First take the adjoining (or next less and next greater) Ratio's in the least Integers to the given one; *viz.* so as the least Term of each Ratio be 1, and the other Term an Integer: then truly compleat the said Ratio's, by annexing to their greater Terms their proper Signs and Fractions: And that Step which hath the greater (secluding their Signs) Fraction thus annex'd to it, must be the 1st, and the other or 2d Step is the I cont. Inc.

Then divide the Fraction in the first Step by that in the I cont. Inc. continuing the Quotient to all the integer Figures thereof, (but here Note, if the Quotient be only 1 +, then this Division may be omitted. The like is to be understood of the following Divisions) by which Figure or Figures multiply the Terms of the I cont. Inc., and to the Products add the respective Terms of the 1st Step, and the Sums thus had are the Terms of the II cont. Inc.

Again, divide the Fraction in the I cont. Inc. by that in the II cont. Inc. continuing, as aforesaid, the Quotient to all the integer Figures thereof, by which (Fig.) multiply the Terms of the II cont. Inc., and to the Products add the respective Terms of the I cont. Inc., and the Sums thus had, are the Terms of the III cont. Inc.

Again, divide the Fraction in the II cont. Inc. by that in the III &c.

And the integer Parts of the cont. Inc. thus found, are the Ratio's, which, with the least whole Numbers, are nearest the Truth; and by which any intermediate Ratio may be found in such Manner, as will be hereafter shewn. So, in relation to this *Proposition*, the adjoining Ratio's in the least Integers to the given one are 1 .. 4 and 1 .. 3, which being truly compleated, are 1 .. 4 — .85840 &c. and 1 .. 3 + .14159 &c.

Operation.

Operation.

	1	1 ·· 4 — .8584073464102 +	
	2	1 ·· 3 + .1415926535898 —	I. cont. Inc.
	3	.14159 &c.) .85840 &c. (6 +	
$2 \times \overline{6}$	4	6 ·· 18 + .8495559215388	
4 + 1	5	7 ·· 22 — .0088514248714	II cont. Inc.
	6	.00885 &c.) .14159 &c. (15 +	
$5 \times \overline{15}$	7	105 ·· 330 — .1327713730710	
7 + 2	8	106 ·· 333 + .0088212805188	III. cont. Inc.
8 + 5	9	*113 ·· 355 — .0000301443526	IV. cont. Inc.
	10	.000030144&c.).0088212 8&c.(292 +	
$9 \times \overline{292}$	11	32996 ·· 103660 — .0088021509592	
11 + 8	12	33102 ·· 103993 + .0000191295596	V cont. Inc.

whose integer Terms consist of eleven Figures, and therefore exceed the Limits prescrib'd by the *Proposition.*

That is to say the Diameter of any Circle
Is to its Circumference nearly As 1 to 4; or nearer as 1 to 3;
Or nearer as 7 to 22, which is *Archimedes's* Ratio;
Or nearer as 106 to 333;
Or nearer as 113 to 355, which is *Metius's* Ratio;
Or nearer as 33102 to 103993, which is nearer the Truth, and better than the Ratio required.

Now there being other intermediate Steps, in each of the one half of which the Ratio of the integer Parts approaches nearer the Truth than that of the integer Parts in the next preceding cont. Inc. (So 6 ·· 19, 5 ·· 16, or 4 ·· 13 is nearer the Truth than 1 ·· 3; &c.) 'tis very probable that some Ratio, consisting only of ten Figures, approaches nearer the Truth than that of *Metius's*, which otherwise wou'd be the Ratio, or Answer required. And for as much as (See the 8th and 9th Steps) 106 continually increas'd with 113, will amount to five Figures before 333 + the same Number of Times increas'd with 355 — will amount to six Figures; 'tis certain that the Terms of the Ratio required consist of five Figures each: The next

* *Here I don't divide the Fraction in the II cont. Inc. by that in the III, the Quotient being only 1 +, and 5 × 1 = 5th Step.*

Thing

Thing to be done therefore is to find how many whole Number of Times 355 — must be added to 333 +, in Order to give a Number the nearest, but not exceeding the greatest one expressible by five Figures, *viz.* 99999, which will be found to be 280 times (thus 99999 — 333 — = 99666 — 5 and 99666 — ÷ 355 — = 280 +); wherefore multiplying the 9th Step by 280 we have 31640 ·· 99400 —, which added to the respective Terms of the 8th Step, give 31746 ·· 99733 +, which 31746 ·· 99733 consisting only of ten Figures, and I say, nearer the Truth than *Metius*'s, is therefore the Ratio sought.

For the last Term thereof divided by the first, quotes 3.141592 64 +, and 355 ÷ 113 = 3.1415929 +: But the former Quotient is nearer the Truth than the latter; therefore &c *w. w. D.*

But the most convenient for Practice, in Cases not requiring great Exactness, is *Metius*'s Ratio, the next subsequent Division, giving 292 + for the Quotient, which is more than any other Quotient in the whole Operation.

PROP. II.

Let it be required to find the Ratio's which, with the least integer Terms, are nearest to that of 2684769 to 8376571.

The adjoining Ratio's to the given one in the least Integers are 1 ·· 4 and 1 ·· 3 (for 8376571 ÷ 2684769 = $3\frac{322264}{2684769}$); both which being truly compleated are 1 ·· 4 — $\frac{2368505}{2684769}$, and 1 ·· 3 + $\frac{322264}{2684769}$.

Now I proceed with the Operation, * omitting to write the common Denominator 2684769, but inserting the Numerators in small Figures, to distinguish them from the preceding Integers.

* *Note, This Method I had from Mr.* Thomas Wallis, *a young Gentleman of a brigbt Genius.*

Operation.

Operation.

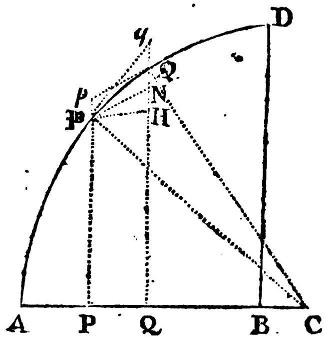
	1	1 ..	4	—	2362505
	2	1 ..	3	+	322264
	3	322 &c.) 2362 &c. (7 +			
$2\times\overline{7}$	4	7 ..	21	+	2255848
$4+1$	5	8 ..	25	—	106657
	6	106 &c.) 322 &c. (3 +			
$5\times\overline{3}$	7	24 ..	75	—	319971
$7+2$	8	25 ..	78	+	2293
	9	2293) 106657 (46 +			
$8\times\overline{46}$	10	1150 ..	3588	+	105478
$10+5$	11	1158 ..	3613	—	1179
$11+8$	12	* 1183 ..	3691	+	1114
$12+11$	13	2341 ..	7304	—	65
	14	65) 1114 (17 +			
$13\times\overline{17}$	15	39797 ..	124168	—	1105
$15+12$	16	40980 ..	127859	+	9
	17	9) 65 (7 +			
$16\times\overline{7}$	18	286860 ..	895013	+	63
$18+13$	19	289201 ..	902317	—	2
	20	2) 9 (4 +			
$19\times\overline{4}$	21	1156804 ..	3609268	—	8
$21+16$	22	1197784 ..	3737127	+	1
	23	1) 2 (2			
$22\times\overline{2}$	24	2395568 ..	7474254	+	2
$24+19$	25	2684769 ..	8376571	—	0

The several Ratio's exhibited by the integer Numbers in the 2, 5, 8, 11, 12, 13, 16, 19, 22 and 25 Steps, are the Ratio's required; of all which, that in the 8th Step, *viz.* 25 to 78, is the most convenient, in Cases not requiring great Nicety, the next succeeding Division producing more for the Quotient than any other in the whole Operation.

* *Here I don't divide the Fraction in the 8th Step by that in the 11th, the Quotient being only 1 +, and* 11 $\times\ \overline{1}$ = 11*th Step.*

PROP.

PROP. IV.



Given the Diameter $= 2$ AC $= d$ of a Circle, as also the Versed-Sine AB $= v$ of the Arc AD. Required to find the Length of the Arc AD.

Suppose the Versed-Sine AB to be divided into an indefinite Number of equal Parts, and each Part $=$ PQ $= a$: thro' any two Points Q and P of which Divisions, at the Distances $z\,a =$ AQ and $:z-1:\times a =$ AP from A, draw the Lines QHN$\mathfrak{Q}q$ and P$\mathfrak{P}p$ $\perp$s AB. Draw $\mathfrak{P}q$ so as to touch the Circle in $\mathfrak{P}$ and intersect Qq in q. Draw $\mathfrak{Q}p$ so as to touch the Circle in $\mathfrak{Q}$ and cut Pp in p; and draw $\mathfrak{P}$N $\parallel$ $p\mathfrak{Q}$ and cutting Qq in N: Draw $\mathfrak{P}$H $\parallel$ AC and cutting Qq in H; and lastly draw C$\mathfrak{Q}$ and C$\mathfrak{P}$. Then it is plain that the indefinitely short circumscribing Line $\mathfrak{P}q$, is greater than the Tangent of its respective Arc, and consequently greater or longer than the said Arc $\mathfrak{P}\mathfrak{Q}$; and that the indefinitely short inscribed Line $\mathfrak{P}$N is less than the Sine of the said Arc $\mathfrak{P}\mathfrak{Q}$, and consequently less or shorter than the aforesaid Arc.

Solution.

The $\llcorner$d $\triangle$s C$\mathfrak{Q}$Q and N$\mathfrak{P}$H are similar; for the $\angle$ C$\mathfrak{Q}$Q $=$ Complment of $\angle$ Q$\mathfrak{Q}p$, and consequently of $\angle$ HN$\mathfrak{P}$ to a $\llcorner$ is $= \angle$ H$\mathfrak{P}$N; therefore $\sqrt{:za\times\overline{d-za}:}$ (Q$\mathfrak{Q}$) $\cdot\cdot$ $\frac{d}{2}$ (C$\mathfrak{Q}$) $::$ a ($\mathfrak{P}$H) $\cdot\cdot$ $\frac{da}{2\sqrt{:dza-zzaa:}} = \mathfrak{P}$N: As also the $\llcorner$d $\triangle$s C$\mathfrak{P}$P and $q\mathfrak{P}$H are similar; for $\angle$ $q\mathfrak{P}$H $=$ Complement of $\angle$ H$\mathfrak{P}$C to a $\llcorner$ is $= \angle$ C$\mathfrak{P}$P; Conseq. $\sqrt{:d\times\overline{z-1}\times a-\overline{z-1}|^2 a^2}:$ (P$\mathfrak{P}$) $\cdot\cdot$ $\frac{d}{2}$ ($\mathfrak{P}$C) $::$ a ($\mathfrak{P}$H) $\cdot\cdot$ $\frac{da}{2\sqrt{:d\times\overline{z-1}\times a-\overline{z-1}|^2 a^2:}} = \mathfrak{P}q$; wheref. (by our *Scholium* 4.) $\frac{da}{2}\times:\overline{da-a^2}|^{-\frac{1}{2}}+\overline{2da-2^2a^2}|^{-\frac{1}{2}}$

+

$+\overline{3da-3^2a^2}|^{-\frac{1}{2}}+$ &c. to $\frac{v}{a}$ Places continued : is $=$ Sum of the inscribed Lines, and therefore $\sqsupset$ Arc AD.

But $\frac{da}{2}\times:\overline{0}|^{-\frac{1}{2}}+\overline{da-a^2}|^{-\frac{1}{2}}+\overline{2da-2^2a^2}|^{-\frac{1}{2}}+$ &c. to $\frac{v}{a}$ Places continued : $=$ Sum of circumscribing Lines is infinite * because of the first Place or Term thereof, *viz.* $\frac{da}{2}\times\overline{0}|^{-\frac{1}{2}}=$ the Length of the first circumscribing Line or Tangent at the Point A, which is manifestly infinite.

* Note, *If the annex'd Fig. were inscribed and circumscribed with Lines pursuant to the above Directions, beginning at the Point* A, *after the Manner practis'd in the first Part of this Chap. it would make the Business here treated of much more intelligible.*

Now the Number of Places in each Series being $=\frac{v}{a}$, and the 1st, 2d, 3d, 4th, &c. Places of the former Series (*viz.* of that expressing the Sum of the inscribed Lines) being the same with the 2d, 3d, 4th, 5th, &c. Places of the latter (*viz.* of the Series expressing the Sum of the circumscribing Lines); This Series therefore, exclusive of the 1st Place or Term thereof, is manifestly less than the former Series, by the last, or least Place thereof: But if to the Series expressing the Sum of the circumscribing Lines, exclusive of the first Term thereof, you add any indefinitely small Quantity greater than the first or greatest Arc (*viz.* the indefinitely small Arc at A), the Sum thence arising will be $\sqsubset$ Arc AD: Whence we may conclude that the above Series expressing the Sum of the inscribed Lines is $=$ Arc AD: But since we have not any Way of finding the Sum of the said former Series in finite Terms, we must have recourse to Approximations by an infinite Series; thus

$$PN=\frac{da}{2}\times\overline{dza-z^2a^2}|^{-\frac{1}{2}}=\frac{\sqrt{da}}{2}\times\overline{z-\frac{z^2a}{d}}|^{-\frac{1}{2}}$$ is (by Sir *If. Newton*'s Theorem) $=\frac{\sqrt{da}}{2}\times:z^{-\frac{1}{2}}+\frac{a\sqrt{z}}{2d}+\frac{3a^2\sqrt{z^3}}{8dd}+\frac{5a^3\sqrt{z^5}}{16d^3}+$ &c. *in infinitum*; Wherefore (by *Scholium* 4.)

$\frac{\sqrt{da}}{2}$

$$\frac{\sqrt{da}}{2} \times : 1^{-\frac{1}{2}} + \frac{a\sqrt{1}}{2d} + \frac{3aa\sqrt{1^3}}{8dd} + \frac{5a^3\sqrt{1^5}}{16d^3} + \&c. :$$

$$+ \frac{\sqrt{da}}{2} \times : 2^{-\frac{1}{2}} + \frac{a\sqrt{2}}{2d} + \frac{3aa\sqrt{2^3}}{8dd} + \frac{5a^3\sqrt{2^5}}{16d^3} + \&c. :$$

$$+ \frac{\sqrt{da}}{2} \times : 3^{-\frac{1}{2}} + \frac{a\sqrt{3}}{2d} + \frac{3aa\sqrt{3^3}}{8dd} + \frac{5a^3\sqrt{3^5}}{16d^3} + \&c. :$$

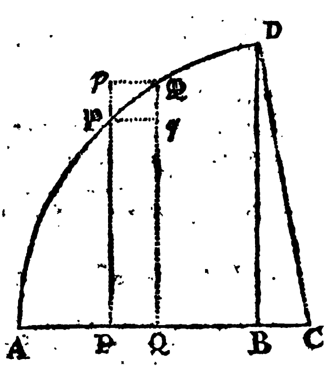
$+$ &c. to $\frac{v}{a}$ Series continued.

$$= w + \frac{\sqrt{v^3}}{6\sqrt{d}} + \frac{3\sqrt{v^5}}{40\sqrt{d^3}} + \frac{5\sqrt{v^7}}{112\sqrt{d^5}} + \&c. \textit{ sine fine}$$

(by our *Lemma* 2. and putting $\frac{\sqrt{da}}{2} \times : 1^{-\frac{1}{2}} + 2^{-\frac{1}{2}} + 3^{-\frac{1}{2}} +$ &c. to $\frac{v}{a}$ Terms continued : equal to w) is $=$ Sum of the inscribed Lines, and is therefore $=$ Arc A D.

Now since such a Series as the last, whose Sum is suppos'd to be $= w$, has not been hitherto consider'd in this Treatise ; 'tis to be presum'd that we know not yet how to find the Value of w : Let us therefore leave this Solution as it is imperfect, and find the Length of the Arc A D otherwise, &c. as in the two following *Scholia*. In Order to which we will premise this

PROP. V.



Given the Versed-Sine AB $= v$ of the Arc AD of a Circle, as also the Diameter $=$ a AC $= d$; Required to find the Area of the Semi-Segment A D B $=$ S.

Suppose the Versed-Sine A B to be divided into an indefinite Number of equal Parts, and each Part $=$ PQ $= a$. Thro' any two Points Q and P of which Divisions, at the Distances $z\,a =$ A Q and $: z - 1 : \times a =$ A P from A, draw the Lines Q q Q and P P p ⊥s AB, and intersecting the Arc A D in the Points Q and P ; and draw the Lines P q and p Q ∥s A B.

 Solution

Solution.

By the Property of a Circle $\sqrt{:za \times \overline{d - za}:} = Q\mathfrak{Q} = \sqrt{:dza - z^2 a^2:}$; and $\sqrt{:d \times \overline{z-1} \times a - \overline{z-1}|^2 a^2:} = P\mathfrak{P}$:

Consequently $a\sqrt{:dza - z^2 a^2:}$ is = Area of the circumscribing ▭ $Pp\mathfrak{Q}Q$; and $a\sqrt{:d \times \overline{z-1} \times a - \overline{z-1}|^2 a^2:}$ is = Area of the inscribed ▭ $P\mathfrak{P}qQ$:

Wherefore (by *Schol.* 4.) $a \times : \sqrt{da - a^2} + \sqrt{2da - 2^2 a^2} + \sqrt{3da - 3^2 a^2}$ + &c. to $\frac{v}{a}$ Places continued: is = Sum of the circumscribing ▭s, and is therefore ⊏ 𝔄.

And $a \times : \sqrt{0} + \sqrt{da - a^2} + \sqrt{2da - 2^2 a^2}$ + &c. to $\frac{v}{a}$ Places continued: is = Sum of the inscribed ▭s; and is therefore ⊐ 𝔄.

Whence the Sum of the circumscribing ▭s exceeds that of the inscribed ▭s only by the Area of the greatest of the former, *viz.* by the indefinitely small Quantity $a\sqrt{:dv - vv:}$; wherefore the Sum of either Series is indefinitely near = 𝔄: But since we have not any Method of finding the Sum of either Series in finite Terms, we must have recourse to Approximations by an infinite Series; thus,

The Area of the circumscribing ▭ $Pp\mathfrak{Q}Q = a\sqrt{:dza - z^2 a^2:}$ is (by Sir *Is. Newton*'s Theorem) $= a \times : \sqrt{dza} - \frac{\sqrt{z^3 a^3}}{2\sqrt{d}} - \frac{\sqrt{z^5 a^5}}{8\sqrt{d^3}} - \frac{\sqrt{z^7 a^7}}{16\sqrt{d^5}} - \frac{5\sqrt{z^9 a^9}}{128\sqrt{d^7}}$ — &c: $= a\sqrt{da} \times : \sqrt{z} - \frac{a\sqrt{z^3}}{2d} - \frac{a^2\sqrt{z^5}}{8d^2} - \frac{a^3\sqrt{z^7}}{16d^3} - \frac{5a^4\sqrt{z^9}}{128d^4}$ — &c. *sine fine*: ; Wherefore (by *Scholium* 4.)

$$a\sqrt{da} \times : \sqrt{1} - \frac{a\sqrt{1^3}}{2d} - \frac{a^2\sqrt{1^5}}{8d^2} - \frac{a^3\sqrt{1^7}}{16d^3} - \frac{5a^4\sqrt{1^9}}{128d^4} - \&c:$$

$$+ a\sqrt{da} \times : \sqrt{2} - \frac{a\sqrt{2^3}}{2d} - \frac{a^2\sqrt{2^5}}{8dd} - \frac{a^3\sqrt{2^7}}{16d^3} - \frac{5a^4\sqrt{2^9}}{128d^4} - \&c:$$

$$+ a\sqrt{da} \times : \sqrt{3} - \frac{a\sqrt{3^3}}{2d} - \frac{a^2\sqrt{3^5}}{8d^2} - \frac{a^3\sqrt{3^7}}{16d^3} - \frac{5a^4\sqrt{3^9}}{128d^4} - \&c:$$

+ &c. to $\frac{v}{a}$ Series continued

$= \frac{2\sqrt{dv^3}}{3} \ominus - \frac{\sqrt{v^5}}{5\sqrt{d}} \ominus - \frac{\sqrt{v^7}}{28\sqrt{d^3}} \ominus - \frac{\sqrt{v^9}}{72\sqrt{d^5}} \ominus - \frac{5\sqrt{v^{11}}}{704\sqrt{d^7}} \ominus$ — &c. *sine fine* (by our *Lemma* 2.) is = Sum of the circumscribing ▭s, and therefore ⊏ $\mathfrak{A}$.

And the Sum of the precedent Series, excepting the last Series; *viz.* the said Sum to $\frac{v}{a} - 1$ Series continued, is (by our *Lemma* 2.) $= \frac{2\sqrt{dv^3}}{3} \ominus - \frac{\sqrt{v^5}}{5\sqrt{d}} \ominus - \frac{\sqrt{v^7}}{28\sqrt{d^3}} \ominus - \frac{\sqrt{v^9}}{72\sqrt{d^5}} \ominus - \frac{5\sqrt{v^{11}}}{704\sqrt{d^7}} \ominus$ — &c. *sine fine*, which is = Sum of the inscribed ▭s, and consequently is ⊐ $\mathfrak{A}$: Therefore, by our *Lemma* 1. $\mathfrak{A} = \sqrt{dv} \times : \frac{2v}{3} - \frac{v^2}{5d} - \frac{v^3}{28dd} - \frac{v^4}{72d^3} - \frac{5v^5}{704d^4} - \frac{7v^6}{1664d^5}$ — &c. *sine fine.*

That is to say, if you reduce $\sqrt{:dv - vv:}$ to an infinite Series, and multiply each Member of that Series by v, and then divide each Member of the Product by the Index of v in that Member, you will have the Value of $\mathfrak{A}$ required.

Scholium I.

Hence the Versed-Sine of any Arc of a Circle along with its Diameter being given, the Length of the Arc itself may be found; thus

First, Drawing the Radius CD [See the precedent Fig.] the Area of the ∟d △ BCD is $= : \frac{d}{4} - \frac{v}{2} : \times \sqrt{:dv - vv:}$ (= $\frac{1}{2}$ CB × BD)

$= \frac{\sqrt{d^3 v}}{4} - \frac{\sqrt{dv^3}}{8} - \frac{\sqrt{v^5}}{32\sqrt{d}} - \frac{\sqrt{v^7}}{64\sqrt{d^3}} - \frac{5\sqrt{v^9}}{512\sqrt{d^5}}$ — &c. *sin. fin.*

$- \frac{\sqrt{dv^3}}{2} + \frac{\sqrt{v^5}}{4\sqrt{d}} + \frac{\sqrt{v^7}}{16\sqrt{d^3}} + \frac{\sqrt{v^9}}{32\sqrt{d^5}}$ + &c. *sin. fin.*

$= \frac{\sqrt{d^3 v}}{4} - \frac{5\sqrt{dv^3}}{8} + \frac{7\sqrt{v^5}}{32\sqrt{d}} + \frac{3\sqrt{v^7}}{64\sqrt{d^3}} + \frac{11\sqrt{v^9}}{512\sqrt{d^5}}$ + &c. *sin. fin.*

This

This added to the precedent Area of the Semi-Segment ADB gives $\frac{\sqrt{d^3 v}}{4} + \frac{\sqrt{d v^3}}{24} + \frac{3\sqrt{v^5}}{160\sqrt{d}} + \frac{5\sqrt{v^7}}{448\sqrt{d^3}} + \frac{35\sqrt{v^9}}{4608\sqrt{d^5}}$ + &c. *sine fine* = Area of the Sector ADC; and this divided by $\frac{d}{4}$ yields $\sqrt{dv} + \frac{\sqrt{v^3}}{6\sqrt{d}} + \frac{3\sqrt{v^5}}{40\sqrt{d^3}} + \frac{5\sqrt{v^7}}{112\sqrt{d^5}} + \frac{35\sqrt{v^9}}{1152\sqrt{d^7}}$ + &c. *sine fine* = the Length of the Arc AD = $\sqrt{dv} \times : 1 + \frac{v}{6d} + \frac{3vv}{40dd} + \frac{5v^3}{112d^3} + \frac{35v^4}{1152d^4}$ + &c. *sine fine* :.

Scholium II.

Now the 2d, 3d, 4th, &c. Terms of the Series in *Prop.* IV. expressing the Sum of the inscribed Lines in the Arc AD, exclusive of the ⊸s, being the same with the respective Terms in the foregoing *Scholium* 1. expressing the Length of the said Arc AD, and, withal the former Series being only = the Sum of the said inscribed Lines; the first Term thereof must therefore be, by an indefinitely small Quantity exceeding the Sum of all the succeeding ⊸s in that Series, less than the first Term of the Series which expresses the Length of the Arc AD: Consequently w (in *Prop.* IV. which see) is $= \sqrt{dv}$ ⟜; ∴ $\frac{\overline{da}|^{\frac{1}{2}}}{2} \times : 1^{-\frac{1}{2}} + 2^{-\frac{1}{2}} + 3^{-\frac{1}{2}}$ + &c. to $\frac{v}{a}$ Terms continued:

$= w$ is $= \overline{dv}|^{\frac{1}{2}}$ ⟜ $= \frac{\overline{da}|^{\frac{1}{2}}}{2} \times \frac{\overline{\frac{v}{a}}|^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$ ⟜: Wherefore, putting n = an indefinite Number, and $p = -\frac{1}{2}$ the Sum of $1^p, 2^p, 3^p, 4^p, 5^p$, &c. to n Terms continued is $= \frac{n^{p+1}}{p+1}$ ⟜.

Now this Canon, with the Difference only of annexing ⊸ instead of ⟜ to it, being = the Sum of the next preceeding Series, when p is = any affirmative Number whatever; as also the said Canon without either ⊸ or ⟜ being = the Sum of the said Series, when p is $= 0$, and the aforesaid Canon being

being $=$ the Sum of the ſaid Series, when p is $= -\frac{1}{2}$; it is manifeſt that the ſaid Canon, *viz.* $\frac{n^{p+1}}{p+1} \infty$ muſt be $=$ the Sum of the ſaid Series, when p is $=$ any intermediate Number between 0 and $-\frac{1}{2}$, as $-\frac{1}{3}$, $-\frac{1}{4}$, $-\frac{1}{5}$, or $-\frac{2}{5}$, $-\frac{3}{7}$, &c.

Corollary.

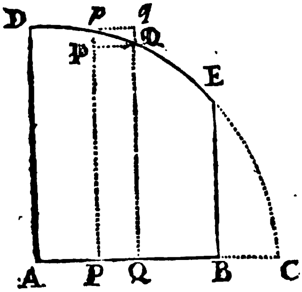
I add further, you may conclude from what hath been ſaid here (there being now no Pretence of Reaſon why we ſhou'd doubt it) that the ſaid Canon $\frac{n^{p+1}}{p+1} \infty$ is $=$ Sum of the ſaid Series, when p is $=$ any negative Number greater than -1.

Scholium III.

v and u being equal to the Verſed-Sines of any two indefinitely ſmall Arcs of the ſame Circle, the Areas of the correſpondent Semi-Segments are, by the precedent *Prop.* and our Method of Notation, equal to $\frac{2v}{3}\sqrt{dv}\,\infty$ and $\frac{2u}{3}\sqrt{du}\,\infty$ reſpectively; and the Lengths of the Arcs themſelves are, by *Schol.* 1. equal to $\sqrt{dv}\,\infty$ and $\sqrt{du}\,\infty$ reſpectively: But $\left(\frac{2v}{3}\sqrt{dv}\,\infty \,..\, \frac{2u}{3}\sqrt{du}\,\infty :: \right) \frac{4v}{3}\sqrt{dv}\,\infty \,..\, \frac{4u}{3}\sqrt{du}\,\infty :: \overline{\sqrt{dv}\,\infty}\,|^3 \,..\, \overline{\sqrt{du}\,\infty}\,|^3 :: dv\sqrt{dv}\,\infty \,..\, du\sqrt{du}\,\infty$; that is to ſay, * the Areas of the indefinitely little Segments, of the ſame Circle ϕ, are in Proportion to one another, as the Cubes of their indefinitely ſmall Arcs.

PROP.

PROP. VI.



Given the Radius A D of the Quadrant A C D, as also A B, and consequently B E being ⊥ A B: Required to find the Area of the Part A B E D of the Quadrant.

$AD\ (=AC) = r$ } given.
$AB = y$ }

Area of the Space A B E D = $\mathfrak{A}$ required.

Suppose the Line A B to be divided into an indefinite Number of equal Parts, and each Part $= a = PQ$; thro' any two Points of those Divisions as Q and P at the Distances $za = AQ$ and $:z-1:\times a = AP$ from A, draw the Lines $Q\mathfrak{Q}q$, and $P\mathfrak{P}p$ ⊥s A B, and intersecting the Arc D E in $\mathfrak{Q}$ and p, and draw the Lines $\mathfrak{P}\mathfrak{Q}$ and pq ||s A B.

Solution.

By the Property of a Circle $\sqrt{:}\ \overline{r+za}\times\overline{r-za}: = \sqrt{:}rr - z^2a^2: = Q\mathfrak{Q}$:

And $\sqrt{:}\overline{r+\overline{z-1}\times a}\times\overline{r-\overline{z-1}\times a}: = \sqrt{:}rr-\overline{z-1}|^2a^2: = Pp$.

Wherefore $a\sqrt{:}rr-z^2a^2:$ is = Area of the inscribed ▭ $PQ\mathfrak{Q}\mathfrak{P}$;

And $a\sqrt{:}rr-\overline{z-1}|^2a^2:$ is = Area of the circumscribing ▭ $PQqp$.

Wherefore, by *Schol.* 4.

$a\times:\sqrt{r^2-a^2}+\sqrt{r^2-2^2a^2}+\sqrt{r^2-3^2a^2}+$ &c. to $\frac{y}{a}$ Places continued: is = Sum of the Areas of the inscribed ▭s, and therefore $\sqsupset\mathfrak{A}$.

And $a\times:\sqrt{r^2}+\sqrt{r^2-a^2}+\sqrt{r^2-2^2a^2}+$ &c. to $\frac{y}{a}$ Places continued: is = Sum of the Areas of the circumscribing ▭s, and therefore $\sqsubset\mathfrak{A}$.

Hence

Hence 'tis manifest that the Sum of the circumscribing ▭s exceeds that of the inscribed ▭s only by the indefinitely little Quantity $ar - a\sqrt{:rr - yy:}$; Consequently the Sum of either Series is indefinitely near $=$ 𝔄: But, since the Sum of either Series cannot be had in finite Terms, we must have recourse to Approximations: thus,

$a\sqrt{:r^2 - z^2a^2:}$ is (by Sir *Is. Newton*'s Theorem) $= a\times : r - \frac{z^2a^2}{2r} - \frac{z^4a^4}{8r^3} - \frac{z^6a^6}{16r^5}$ — &c. *sine fine*: ; Wherefore (by our *Scholium* 4.)

$$a\times : r - \frac{a^2}{2r} - \frac{a^4}{8r^3} - \frac{a^6}{16r^5} \text{ — \&c. in infinitum :}$$

$$+ a\times : r - \frac{2^2a^2}{2r} - \frac{2^4a^4}{8r^3} - \frac{2^6a^6}{16r^5} \text{ — \&c. in infinitum :}$$

$$+ a\times : r - \frac{3^2a^2}{2r} - \frac{3^4a^4}{8r^3} - \frac{3^6a^6}{16r^5} \text{ — \&c. in infinitum :}$$

$+$ &c. to $\frac{y}{a}$ Series continued

$$= ry - \frac{y^3}{6r}\circ - \frac{y^5}{40r^3}\circ - \frac{y^7}{112r^5}\circ \text{ — \&c. in infinitum,}$$

by our *Lemma* 2. is $=$ the Sum of the Areas of the inscribed ▭s, and therefore ⊐ 𝔄.

And the Sum of the foregoing Series $+ ar - a\sqrt{:rr - yy:}$

$$= ry - \frac{y^3}{6r}\circ - \frac{y^5}{40r^3}\circ - \frac{y^7}{112r^5}\circ, \text{ \&c. in infinitum is}$$

$=$ Sum of the Areas of the circumscribing ▭s, and therefore ⊏ 𝔄.

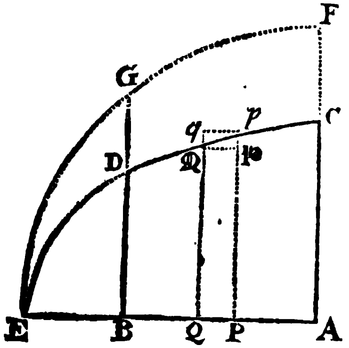
Consequently (by our *Lemma* 1.) $ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} - \frac{5y^9}{1152r^7}$ — &c. *sine fine* is $=$ 𝔄.

That is to say, If you reduce $\sqrt{:rr - yy:}$ to an endless Series, and multiply each Member of that Series by y, and then divide each Member of the Product by the Index of y in that Member, you will have the Value of 𝔄 required.

Note, *By the Help of this* Prop. VI. *the Ratio of the Diameter of a Circle to its Circumference may be found very near the Truth.*

*O PROP.

PROP. VII.



Given the Semi-conjugate Diameter AC, as also the Part AB of the given Semi-transverse Diameter AE, and consequently BD being || AC: Required to find the Area of the Portion ABDC of the Ellipsis.

$$\left.\begin{array}{l} AE = r \\ AB = y \\ AC = c \end{array}\right\}\text{given.}$$

Area of the Part ACDB of the Ellipse $= \mathfrak{A}$ required.

Suppose the Line AB to be divided into an indefinite Number of equal Parts, and each Part $= a =$ PQ; and at the indetermined Distances $za =$ AQ, and $:z-1:\times a =$ AP from A erect the Lines $Q\mathfrak{Q}q$ and $P\mathfrak{P}p$ ⊥s AB, and intersecting the Curve CD in $\mathfrak{Q}$ and p; and draw the Lines qp and $\mathfrak{Q}\mathfrak{P}$ || s BA.

Solution.

By the Property of an Ellipse r(EA) ·· c(AC) :: $\sqrt{:}\overline{r+za}\times\overline{r-za}:$ $(=\sqrt{:}\overline{AE+AQ}\times QE:)$ ·· $\frac{c}{r}\sqrt{:r^2-z^2a^2:}$ $= Q\mathfrak{Q}$ :: $\sqrt{:}\overline{r+\overline{z-1}\times a}\times\overline{r-\overline{z-1}\times a}:$ ·· $\frac{c}{r}\sqrt{:r^2-\overline{z-1}|^2a^2:} = Pp:$

Wherefore $\frac{ca}{r}\sqrt{:r^2-z^2a^2:}$ is = Area of the ▭ $PQ\mathfrak{Q}\mathfrak{P}$;

And $\frac{ca}{r}\sqrt{:rr-\overline{z-1}|^2a^2:}$ is = Area of the ▭ $PQqp$:

Wherefore it is plain from the next preceeding *Probl.* that $\frac{c}{r}\times:ry-\frac{y^3}{6r}-\frac{y^5}{40r^3}-\frac{y^7}{112r^5}-\frac{5y^9}{1152r^7}-\frac{7y^{11}}{2816r^9}-$ &c. *sine fine*: is $= \mathfrak{A}$.

Corol.

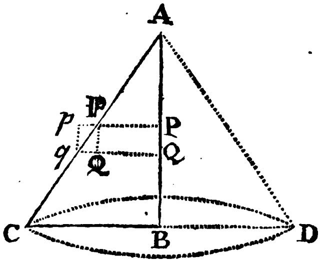
Corollary.

$$2c \cdot\cdot 2r :: \frac{c}{r}\times : ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} - \&c. : \cdot\cdot$$

$ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5}$, &c; that is (by the two foregoing *Propositions*)

As the Conjugate-Diameter is to the Transverse-Diameter, So is the Area of the Portion ACDB of the Ellipsis to that of the respective Portion AFGB of the Circle describ'd by the Semi-Transverse-Diameter AE as a Radius; and consequently, so is the Area of the whole Ellipsis to that of the whole Circle.

PROP. VIII.



Given the Axis AB of the Cone ABCD form'd by the Revolution of the ∟d △ ABC upon the said Axis; as also the Radius BC of its Base, and * consequently the Circumference describ'd by the Point C in that Revolution: Required to find the Solidity of the Cone.

$$\left.\begin{array}{r} AB = b, \\ CB = BD = r \\ \text{Circumf. of the Base} = c \end{array}\right\} \text{given.}$$

Solidity of the Cone ABCD $= \mathfrak{S}$ required.

Suppose the Axis AB to be divided into an indefinite Number of equal Parts, and each of those Parts to be $= a =$ PQ; and thro' any two of those Divisions, as Q and P, at the indetermined Distances $za =$ AQ, and $:z-1:\times a =$ AP from A, draw the Lines Q$\mathfrak{Q}q$ and P$\mathfrak{P}p$ ⊥s AB, and intersecting AC in q and $\mathfrak{P}$; and draw the Lines qp and $\mathfrak{Q}\mathfrak{P}$ ||s AB.

Solution.

The ∟d △s ABC, AQq and AP$\mathfrak{P}$ are similar; wheref.

$$b(AB) \cdot\cdot r(BC) :: \left\{\begin{array}{l} za(AQ) \cdot\cdot \dfrac{rza}{b} = Qq \\ :z-1:\times a(AP) \cdot\cdot \dfrac{:z-1:\times ra}{b} = P\mathfrak{P}. \end{array}\right.$$

---

* See Prop. III.

*Oo And

* i. e. *Their Areas.*

And since * Circles are to one another in Proportion as the Squares of their Radii, r^2 ($\overline{CBq}$) .. $\frac{1}{2}rc$ (the Area of the Circle describ'd by the Radius CB in the said Revolution) :: $\frac{z^2a^2r^2}{bb}$ ($\overline{Qq}^2$) .. $\frac{rcz^2a^2}{2bb}$ (Area of the Circle describ'd by the Line Qq in the said Revolution) :: $\frac{\overline{z-1}|^2 a^2 r^2}{bb}$ ($P\mathfrak{P}q$) .. $\frac{rc\times\overline{z-1}|^2 a^2}{2bb}$ = Area of the Circle describ'd by the Line $P\mathfrak{P}$ in the said Revolution : Wherefore $a\times\frac{rcz^2a^2}{2bb}$ is = * solidity of the Cylinder generated by the ▭ $QPpq$ in the said Revolution ; And $a\times\frac{rc\times\overline{z-1}|^2a^2}{2bb}$ is = Solidity of the Cylinder generated by the ▭ $QP\mathfrak{P}\mathfrak{Q}$ in the aforesaid Revolution : Wherefore (by our *Scholium* 4.)

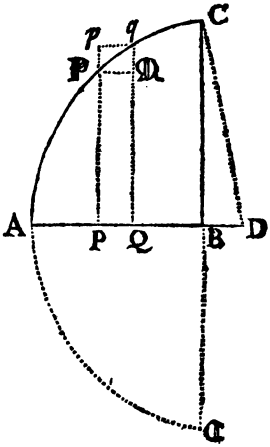
* Note, *I suppose it known that the Solidity of a Cylinder is = Area of its Base* × *its Height.*

$\frac{rca^3}{2bb}\times : 1^2+2^2+3^2+4^2+$ &c. to $\frac{b}{a}$ Terms continued : $=\frac{rcb}{6}+\frac{rca}{4}\phi$ (by our *Lemma* 2.) is = the Sum of Solidities of the indefinitely little circumscribing Cylinders, and therefore ⊏ $\mathfrak{S}$.

And $\frac{rca^3}{2bb}\times : 0+1^2+2^2+3^2+$ &c. to $\frac{b}{a}$ Terms continued : $=\frac{rcb}{6}-\frac{rca}{4}\phi$ is = the Sum of the Solidities of the inscribed indefinitely little Cylinders, and therefore ⊐ $\mathfrak{S}$: Consequently (by our *Lemma* 1.) $\frac{rcb}{6}$ is = $\mathfrak{S}$.

PROP.

PROP. IX.



Given the Versed-Sine, or Altitude A B of the Segment C A ℭ of of a Sphere, the said Segment being form'd by the Revolution of the Semi-Segment B A C of a Circle upon its Semi-Axis A D, as also the Radius A D, and consequently the Circumference of a great Circle of that Sphere: Required to find the Solidity of the said Segment of the Sphere.

$$\left.\begin{array}{r} AB = b \\ AD = r \\ \text{Circumf. of a great Circle of the Sphere} = c \end{array}\right\} \text{given}$$

Solidity of the Segment C A ℭ of the Sphere (generated as aforesaid) $\Big\} = \mathfrak{a}$ sought.

Preparation.

Suppose the Line A B to be divided into an indefinite Number of equal Parts, and each Part $= a =$ P Q; and thro' any two Points Q and P of those Divisions, at the Distances $za = $ A Q and $:z-1:\times a =$ A P from A, draw the Lines Q 𝔔 q and P 𝔓 p ⊥s A B, and intersecting the Arc A C in q and 𝔓; and draw the Lines pq and 𝔓 𝔔 ∥s A B.

Solution.

By the Property of a Circle $: 2r - za : \times za = 2rza - z^2a^2 = \overline{Qqq}$; And $: 2r - \overline{z-1} \times a : \times : z-1 : \times a = 2r \times \overline{z-1} \times a - \overline{z-1}|^2 a^2 =$ P 𝔓 q:

Since Circles are to one another in duplicate Ratio of their Radii; therefore rr (ADq) $\cdot\cdot$ $\frac{1}{2}rc$ (Area of a great Circle of the Sphere) $::$ $2rza - zzaa$ (Qq × Qq) $\cdot\cdot$ $cza - \dfrac{cz^2a^2}{2r} =$ Area of the Circle describ'd by the Line Qq in revolving as aforesaid $::$ $2r \times \overline{z-1} \times a - \overline{z-1}|^2 a^2$ (P 𝔓 q) $\cdot\cdot$

c ×

$c \times \overline{z-1} \times a - \frac{\overline{c \times z - 1}|^2 a^2}{2r}$ = Area of the Circle describ'd by P $\mathfrak{P}$ in the said Revolution; therefore $a \times : cza - \frac{czzaa}{2r}$: is = Solidity of the indefinitely little Cylinder form'd by ▭ QPpq in the said Revolution; And $a \times : c \times \overline{z-1} \times a - \frac{\overline{c \times z - 1}|^2 a^2}{2r}$: is = Solidity of the indefinitely little Cylinder form'd by ▭ QP$\mathfrak{P}\mathfrak{Q}$ in the aforesaid Revolution: Wherefore (by our *Schol.* 4.) $ac \times : \overline{a - \frac{a^2}{2r}} + \overline{2a - \frac{2^2 a^2}{2r}} + \overline{3a - \frac{3^2 a^2}{2r}} + \overline{4a - \frac{4^2 a^2}{2r}}$ + &c. to $\frac{b}{a}$ Places continued: (the two Members under each Line of Connexion being what is here call'd a Place) $= : \frac{bbc}{2} + \frac{bca}{2} : - : \frac{b^3 c}{6r} + \frac{b^2 ca}{4r} \phi$: (by our *Lemma* 2.) $= \frac{bbc}{2} - \frac{b^3 c}{6r} \ominus$ is = the Sum of the Solidities of the circumscribing indefinitely little Cylinders, and therefore $\sqsubset \mathfrak{A}$: And $ac \times : \overline{0 - 0} + \overline{a - \frac{aa}{2r}} + \overline{2a - \frac{2^2 a^2}{2r}} + \overline{3a - \frac{3^2 a^2}{2r}}$ + &c. to $\frac{b}{a}$ Places continued: $= : \frac{bbc}{2} - \frac{bca}{2} : - : \frac{b^3 c}{6r} - \frac{bbca}{4r} \phi : = \frac{bbc}{2} - \frac{b^3 c}{6r} \ominus$ is = the Sum of the Solidities of the indefinitely little inscribed Cylinders; and therefore $\sqsupset \mathfrak{A}$: Whence (by our *Lemma* 1.) $\frac{bbc}{2} - \frac{b^3 c}{6r}$ is $= \mathfrak{A}$.

Scholia.

Scholia.

1. If it were required to find the Solidity of the Figure generated by the Sector D A C, in revolving upon the Semi-Axis D A, it it may be briefly done from the two foregoing Propositions, thus: B A being $= b$, D B is $= r - b$; and, by the Property of a Circle, $\surd : 2rb - bb :$ is $=$ BC: And $r \cdot\cdot c :: \surd : 2rb - bb : \cdot\cdot \frac{c}{r} \surd : 2rb - bb :$; Wherefore (by Prop. VIII.)

$$\frac{: r - b : \times \surd : 2rb - bb : \times \frac{c}{r} \surd : 2rb - bb :}{6} = \frac{2crrb - 3crbb + cb^3}{6r}$$

is $=$ Solidity of the Cone form'd by the Revolution of ∟d △ DCB upon DB as an Axis; consequently, $\frac{bbc}{2} - \frac{b^3c}{6r} + \frac{2crrb - 3crbb + cb^3}{6r} = \frac{crb}{3}$ is $=$ Solidity of the solid Figure generated by the Sector D A C in revolving upon D A as an Axis.

2. Therefore, by * *Pardie*'s Theorem $\frac{crb}{3} \div \frac{r}{3} = cb$ is $=$ Area of the Surface, generated by the Arc A C, in revolving upon D A as an Axis.

* *See* Pardie's *Elem. Prop.* or Sturmius's *Mathe.* Enucl. *Def.* 20. *Consect.* 1.

3. If b be $= 2r$, then $2rc$ will be $=$ the whole Surface of the Sphere: Whence you have the Demonstration of one of the great *Archimedes*'s Theorems; namely, that the Surface of a Sphere is quadruple the Area of its greatest Circle; for $2rc$ (the Area of the Surface of a Sphere) is $= \frac{1}{2} rc$ (the Area of one of its greatest Circles) $\times 4$.

Note, In order to find the Value of any propos'd Figure, you must take care that the indefinitely little Figures be inscribed in, or circumscribed about the propos'd Figure, so as sufficiently to exhaust, or be exhausted by it. Then you need only find the Sum of the Values of either these inscribed or circumscribing indefinitely little Figures; and then rejecting (or leaving out) the indefinitely small Quantities (or the Signs •, •- and ȧ) in either of these Sums, the Remainder will express the Value sought of the propos'd Figure.

PROP.

PROP. X.

If the Abſciſſe of any plane Figure be $= x$, and the Ratio of each Part of that Abſciſſe to its reſpectioe Semi-ordinate, rightly applied, as x to $\overline{x+b}\big|^{\frac{1}{n}}$:

Then 'tis manifeſt, that the Sum of the Areas of the circumſcribing indefinitely little Paralellograms will be found $=$

$$a^{1+\frac{1}{n}} \times : \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{4+\frac{b}{a}}\Big|^{\frac{1}{n}} + \&c.$$

to $\frac{x}{a}$ Places continued : $= a^{1+\frac{1}{n}} \times : 1^{\frac{1}{n}} + 2^{\frac{1}{n}} + 3^{\frac{1}{n}} + 4^{\frac{1}{n}} + \&c.$ to $\frac{x+b}{a}$ Terms continued : $- a^{1+\frac{1}{n}} \times : 1^{\frac{1}{n}} + 2^{\frac{1}{n}} + 3^{\frac{1}{n}} + \&c.$ to $\frac{b}{a}$ Terms continued : $= a^{1+\frac{1}{n}} \times \overline{\frac{x+b}{a}}\Big|^{\frac{1}{n}+1} \bullet \div \overline{\frac{1}{n}+1} - a^{\frac{1}{n}+1} \times \overline{\frac{b}{a}}\Big|^{\frac{1}{n}+1} \bullet \div \overline{\frac{1}{n}+1}$ (by our *Lemma* 2.) $= \frac{n}{n+1} \times \overline{x+b}\big|^{\frac{n+1}{n}} - \frac{n}{n+1} \times \overline{b}\big|^{\frac{n+1}{1}} \bullet$; Wherefore (ſee the preceding Note) $\frac{n}{n+1} \times \overline{x+b}\big|^{\frac{n+1}{n}} - \frac{n}{n+1} \times \overline{b}\big|^{\frac{n+1}{n}}$ is $=$ the Area of the Fig. mentioned in this Prop. X.

PROP. XI.

If the Abſciſſe of any plane Figure be $= x$, and the Ratio of every Part of the Abſciſſe to its reſpective Semi-ordinate, rightly applied, as x to $x \times \overline{x+b}\big|^{\frac{1}{n}}$;

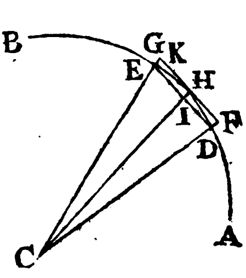
Then the Sum of the Areas of the circumſcribing indefinitely little ▭s will be found $=$

a^2+

$a^{2+\frac{1}{n}} \times : 1 \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + 2 \times \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} + 3 \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}}$ + &c.

to $\frac{x}{a}$ Places continued $: = a^{2+\frac{1}{n}} \times : \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}+1} +$
$\overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}+1} + \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}+1} +$ &c. to $\frac{x}{a}$ Places continued : $(= a^{2+\frac{1}{n}} \times : \overline{1+\frac{b}{a}} \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{2+\frac{b}{a}} \times$
$\overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{3+\frac{b}{a}} \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}} +$ &c. to $\frac{x}{a}$ Places continued :) $- a^{2+\frac{1}{n}} \times : \frac{b}{a} \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + \frac{b}{a} \times \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} +$
$\frac{b}{a} \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}} +$ &c. to $\frac{x}{a}$ Places continued $: =$ (by Prop. X.) $\frac{n}{2n+1} \times \overline{x+b}\Big|^{\frac{2n+1}{n}}$ ⊖ $- \frac{n}{2n+1} \times \overline{b}\Big|^{\frac{2n+1}{n}}$
⊕ $- \frac{n}{n+1} \times b \times \overline{x+b}\Big|^{\frac{n+1}{n}}$ ⊖ $+ \frac{n}{n+1} \times b \times \overline{b}\Big|^{\frac{n+1}{n}}$ ⊕
which, after rejecting the Signs ⊖ and ⊕, is = Area of the Plane Figure mentioned in this *Prop.* XI.

Lemma.



Any indefinitely little Curve DHE is, at most, but by an indefinitely-indefinite small Quantity greater than its Subtense DIE.

Let the Curve, whose indefinitely small Part is supposed to be DHE, be AB; within, or Concave to, any Part of which, at any convenient finite Distance from it, assume the Point C, so as drawing the straight Lines CEG, CDF and CIH; as also the

*Pp

the Line GHF || EID, and so as to touch the Curve in the Point H, that GHF may be ⊏ the Curve EHD (which may be easily done in any Curve, only by making the finite Lines CE and CD short enough). Then it is evident, that since ED is indefinitely little, IH must, * at most, be so too; wherefore (from the similar $\triangle$s CID and CHF, CIE and CHG) DF and EG must be, each of them, at most, but indefinitely little. Now the $\triangle$s CED and CGF are similar; wherefore CD ·· CD + DF :: DIE

*IH *is really indefinitely-indefinite little in all Sorts of Curves that I know.*

$$\cdot\cdot \text{FHG} = \frac{:\text{CD}+\text{DF}:\times\text{DIE}}{\text{CD}} = \text{DIE} + \frac{\text{DF}\times\text{DIE}}{\text{CD}}:$$

And DIE being indefinitely small, and DF, at most, but so too; FHG must therefore exceed DIE but by an indefinitely-indefinite small Quantity at most; But DHE is manifestly greater than DIE, and by Position, less than FHG; wherefore DIE is less, and FHG is greater than the indefinitely small Curve DHE but by an indefinitely-indefinite small Quantity at most. *Q. E. D.*

Corollary.

Hence 'tis evident, that the Curve HE is but an indefinitely-indefinite small Quantity, at most, greater than HK = IE.

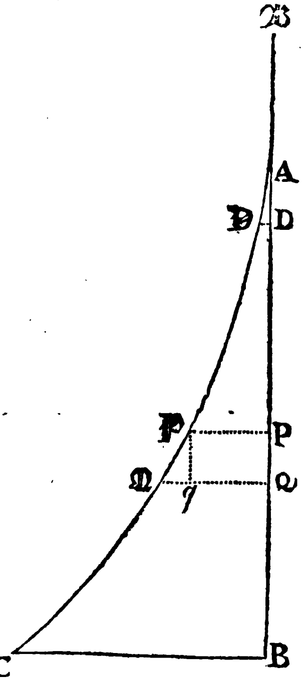
Note, *It would be sufficient for our present Purpose to suppose the Truth of this* Lemma, *and then proceed accordingly to the Solution of the following Proposition, which, when done, we may then apply this* Lemma *to that particular* Curve; *and then, if it succeeds, (as certainly it will) we may conclude the* Proposition *true, and the* Lemma *too in that Particular.*

PROP. XII.

How to find the Nature of a Curve, in which a heavy Body shall, by the Force of its Gravity, descend only equal Spaces in equal Times.

Let

Let $\mathfrak{B}AB$ be a straight Line, perpendicular to the Plane of the Horizon; and let the Curve $A\mathfrak{D}\mathfrak{P}\mathfrak{Q}C$ be the Curve required, whose Axe is in the same straight Line with $\mathfrak{B}B$.



The Body is now supposed to fall from $\mathfrak{B}$ to A, and from thence to commence a Motion in the Curve $A\mathfrak{D}\mathfrak{P}\mathfrak{Q}C$, in which Curve it must still move with the Velocity acquired by the Descent from $\mathfrak{B}$ to A.

Now suppose $\mathfrak{B}A = b$, $AB = x$ and BC (being $\perp AB$) $= y$.

Suppose also the Line AB to be divided into an indefinite Number of equal Parts, and each Part $= a = AD$; then through D, and any other two Points of those Divisions, at the Distances $a = AD$, $:z - 1: \times a = AP$, and $za = AQ$ from A, draw the Lines $D\mathfrak{D}$, $P\mathfrak{P}$ and $Qq\mathfrak{Q}$ $\perp$s AB and $\parallel$s BC, and meeting the Curve in $\mathfrak{D}$, $\mathfrak{P}$ and $\mathfrak{Q}$. Draw also $\mathfrak{P}q \parallel AB$; and draw the Subtense $\mathfrak{P}\mathfrak{Q}$. Then AD, being indefinitely small, is therefore (by the precedent *Corollary*) but by an indefinitely-indefinite small Quantity less than $A\mathfrak{D}$; wherefore $A\mathfrak{D}$ is (by our way of notation) $= a\odot$: Also $\sqrt{:\mathfrak{P}qq + \mathfrak{Q}qq:}$ being $=$ subtense $\mathfrak{P}\mathfrak{Q} = \sqrt{:aa + q\mathfrak{Q}q:}$, the Curve $\mathfrak{P}\mathfrak{Q}$ is therefore (by the foregoing *Lemma*, and by our way of Notation) $= \sqrt{:aa + q\mathfrak{Q}q:}\odot$. Now the Times of Description being universally as the Spaces directly, and the Velocities reciprocally; and the Velocities acquired in the Points A and P being those acquired by the Descents thro' $\mathfrak{B}A$ and $\mathfrak{B}P$, that is, being in a subduplicate Ratio of the Lines of Descent; therefore the Time of describing $A\mathfrak{D}$ is $= \dfrac{A\mathfrak{D}}{\sqrt{\mathfrak{B}D}} = \dfrac{*a}{\sqrt{b}\odot}\odot$, and the Time of describing

* Note, $\dfrac{a}{\sqrt{b}\odot}\odot$ is $= \dfrac{a}{\sqrt{b}}\odot$ because $\dfrac{a}{\sqrt{b}\odot}\odot$ is rather $\dfrac{a}{\sqrt{b}\odot}\odot^2$

$\mathfrak{PQ}$ is $= \frac{\mathfrak{PQ}}{\sqrt{BQ}} = \frac{\sqrt{:aa + q\mathfrak{Q}q:}}{\sqrt{:b+za:}}$ ⊙ : And because, by Hyp. the Curve is suppos'd to be such, that the Body, which describes it, makes equal Descents in equal Times; therefore we have this Equation $\frac{a}{\sqrt{b}} = \sqrt{\frac{aa+q\mathfrak{Q}q}{b+za}}$ ⊙ ; wherefore $baa + b \times q\mathfrak{Q}q$ ⊙ $= baa + za^2$; whence $\sqrt{\frac{aaa}{b}} \times \sqrt{z}$ is $= q\mathfrak{Q}$ ⊙ : Consequently (by our *Scholium* 4.) $\sqrt{\frac{aaa}{b}} \times : \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} +$ &c. to $\frac{x}{a}$ Terms continued $:= \sqrt{\frac{a^2}{b}} \times \overline{\frac{x}{a}}\Big|^{\frac{3}{2}}$ ⊙ $\div \frac{3}{2} = \frac{2}{3}\sqrt{\frac{xxx}{b}}$ ⊙ (by our *Lemma* 2.) is $= y$ ⊙ ; wherefore $\frac{4}{9}x^3$ is $= by^2$, which is the Nature or Property of the Curve sought; and shews that it is what is call'd a Semi-cubical Paraboloid Convex towards the Axis, and its Parameter is $= \frac{9}{4}b$.

Scholium I.

If the Relation between the Curve and Abscisse were required, it may be found thus; $\frac{a}{\sqrt{b}}$ is (by what has been before found) $= \sqrt{\frac{aa+q\mathfrak{Q}q}{b+za}}$ ⊙, that is $= \sqrt{\frac{\text{Curve } \mathfrak{PQ}q}{b+za}}$ ⊙ ; wherefore $\sqrt{\frac{a^2}{b}} \times \sqrt{:z+\frac{b}{a}:}$ ⊙ is $=$ Curve $\mathfrak{PQ}$ ⊙ ; wherefore (by our *Scholium* 4.) $\sqrt{\frac{aaa}{b}} \times : \sqrt{1+\frac{b}{a}} + \sqrt{2+\frac{b}{a}} + \sqrt{3+\frac{b}{a}} + \sqrt{4+\frac{b}{a}} +$ &c. to $\frac{x}{a}$ Places continued $:= \sqrt{\frac{a^2}{b}} \times : \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} +$ &c. to $\frac{x+b}{a}$ Terms continued $: - \sqrt{\frac{a^2}{b}} \times : \sqrt{1} + \sqrt{2} + \sqrt{3} +$ &c. to $\frac{b}{a}$ Terms continued $: = \sqrt{\frac{a^2}{b}} \times \overline{\frac{x+b}{a}}\Big|^{\frac{3}{2}}$ ⊙ $\div \frac{3}{2} - \sqrt{\frac{a^2}{b}} \times \overline{\frac{b}{a}}\Big|^{\frac{3}{2}}$ ⊙ $\div \frac{3}{2}$

(by

(by our *Lemma* 2.) is = Curve A $\mathfrak{D}\,\mathfrak{P}\,\mathfrak{Q}$ C $\bullet$; that is $\frac{2\times\overline{x+b}|^{\frac{3}{2}}}{3\sqrt{b}}-\frac{2}{3}b$ is = Curve A $\mathfrak{D}\,\mathfrak{P}\,\mathfrak{Q}$ C.

Scholium II.

Or the Ratio of the Absciſſe AB to BC being ſuch as was before found, and AB alſo given; the Length of the Curve A $\mathfrak{D}\,\mathfrak{P}\,\mathfrak{Q}$ C may be otherwiſe determined; thus

AP being (by ſuppoſition) $=: z-1 :\times a$, P$\mathfrak{P}$ is therefore (by the before found Ratio) $=\frac{2\times:\overline{z-1:\times a}|^{\frac{3}{2}}}{3\sqrt{b}}$: And AQ being alſo $=za$, Q$\mathfrak{Q}$ is likewiſe $=\frac{2}{3}\sqrt{\frac{z^3a^3}{b}}$; wherefore $q\mathfrak{Q}$ is $=\frac{2}{3}\sqrt{\frac{z^3a^3}{b}}-\frac{2\times:\overline{z-1:\times a}|^{\frac{3}{2}}}{3\sqrt{b}}=\frac{2}{3}\sqrt{\frac{z^3a^3}{b}}$ $-\frac{2}{3\sqrt{b}}\times:\overline{za}|^{\frac{3}{2}}+\frac{3}{2}\times\overline{za}|^{\frac{3}{2}-1}\times-a+\frac{3}{2}\times\frac{\frac{3}{2}-1}{2}\times$ $\overline{za}|^{\frac{3}{2}-2}\times aa\phi:=\sqrt{\frac{za^3}{b}}-\frac{1}{4}\sqrt{\frac{a^3}{bz}}\phi$; wherefore $\mathfrak{P}\mathfrak{Q}$ $=\sqrt{:aa+\frac{za^3-\frac{1}{2}a^3}{b}\phi:}$; that is (becauſe $-\frac{1}{2}a^3\phi$ having no z inſerted in it, is indefinitely little in reſpect of aa, and therefore cannot affect it only by way of $\bullet$) $\mathfrak{P}\mathfrak{Q}$ = $\sqrt{\frac{baa+za^3}{b}}\phi$; wherefore (by *Scholium* 4.) $\sqrt{\frac{a^3}{b}}\times:$ $\sqrt{\frac{b}{a}+1}+\sqrt{\frac{b}{a}+2}+\sqrt{\frac{b}{a}+3}+$ &c. to $\frac{x}{a}$ Places continued: = (manifeſtly) $\sqrt{\frac{aaa}{b}}\times:\sqrt{1}+\sqrt{2}+\sqrt{3}+$ $\sqrt{4}+$ &c. to $\frac{x+b}{a}$ Terms continued: $-\sqrt{\frac{aaa}{b}}\times:\sqrt{1}$ $+\sqrt{2}+\sqrt{3}+$ &c. to $\frac{b}{a}$ Terms continued: $=\sqrt{\frac{a^3}{b}}\times\overline{\frac{x+b}{a}}|^{\frac{3}{2}}$ $\bullet\div\frac{3}{2}-\sqrt{\frac{a^3}{b}}\times\overline{\frac{b}{a}}|^{\frac{3}{2}}\bullet\div\frac{3}{2}$ (by our *Lemma* 2.) is indefi-

nitely

nitely near $=$ Curve A D D D C; that is, the Length of the said Curve is $= \frac{2 \times \overline{x+b}|^{\frac{3}{2}}}{3\sqrt{b}} - \frac{2}{3}b$; as before.

Theorem.

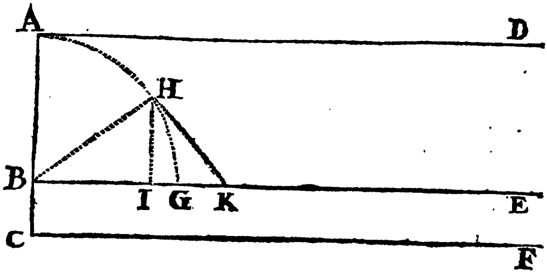
The Sum of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c, *Sine fine* (upon which Series depends the Quadrature of that Hyperbola whose Abſciſſe is to its Ordinate as x to x^{-1}) is indefinitely near $=$ *Neper*'s Legarithm of $\frac{1}{0}$.

Demonſtration.

By *Part* XV. *Chap.* 3. *Rule* 1. the Logarithm of $1-x$ is $= \frac{10000 \text{ \&c. indefinitely}}{n} \times -: x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$ &c. in Infinitum ϕ : ; wherefore *Neper*'s Logarithm of $1-x$ is (becauſe n, in this Caſe, is equal to 10000 &c. indefinitely) $= 1 \times -: x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$ &c. in Infinitum ϕ : ; wherefore *Neper*'s Logarithm of $\frac{1}{1-x}$ is found, by ſubtracting *Neper*'s Logarithm of $1-x$ from his Logarithm of 1, $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. in Infinitum ϕ; conſequently, putting $x = 1$, you have *Neper*'s Logarithm of $\frac{1}{0} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. in Infinitum ϕ. *Q. E. D.*

PROP.

PROP. XIII.



Given the Diſtances AB and BC between the three infinite parallel ſtraight Lines AD, BE and CF, bounded on the left hand at A, B and C, and produc'd infinitely towards the Right: Required to find the Ratio of the Area of the infinite Space ABED to that of the infinite Space BCFE.

On B as a Center with the Radius BA deſcribe the Quadrant BGHA, from any Point H of whoſe Circumference let fall HI ⊥ BE: Draw HK ſo as to touch the Circumference in H, and interſect BE in K; and draw BH. Then the △s BIH and HIK are ſimilar; wherefore, ſuppoſing the Radius $BG = r$, $GI = x$, $IH = y$, and $IK = t$, we have $\frac{yy}{r-x} = t$. But when HI coincides with AB, x, as alſo y becomes $= r$, and $\frac{yy}{r-x} = t = \frac{rr}{r-r} = BE$; conſequently the Area of the infinite Space $ABED = \frac{r^3}{r-r}$. In like manner the Area of the infinite Space BCFE will be found $= \frac{s^3}{s-s}$, putting $BC = s$.

Now when any one tells me the Ratio which once Nothing has to twice Nothing, I think I ſhall be able to reſolve whether $\frac{r^3}{r-r}$ be to $\frac{s^3}{s-s}$ as rr is to ss, and (ſtrictly ſpeaking) not in any other Ratio; or whether $\frac{r^3}{r-r}$ be $= \frac{s^3}{s-s}$; but till then, I will not pretend to determine which of them is the Truth.

Corollary.

Corollary.

1. Hence it is manifeſt, that any finite Quantity whatſoever, added to or taken from an infinite one, does not increaſe or diminiſh the ſaid infinite one: For $\frac{r^3}{r-r}$ is, by Diviſion, found to be $= r^2 + r^2 + r^2 + r^2 +$ &c. $+ \frac{r^3}{r-r}$; conſequently $\frac{r^3}{r-r}$ + or − any finite Number of r^2's is $= \frac{r^3}{r-r}$.

2. As BI, by a continual Sub-diviſion, will never be reduc'd to a Point or a Line of no Length; ſo by a continual Appoſition of finite Spaces an infinite one will never be produc'd: But (ſay you) does not an infinite Number (if it be proper to call it ſo) of finite Spaces produce an infinite one? Yes, and, I ſay, an infinite Number of Nothings may produce ſomething; ſo $0 \times \frac{1}{0}$ is $= 1$. To account for which Paradoxes take this parallel Caſe.

If one ſay ſuch a thing will happen in an infinite Number of Days to be reckon'd from hence; I ſay his Affirmation is only negative, that is, he, in effect, ſays that it will never happen: For it is plain that that thing is to happen when a Line bounded, ſuppoſe, on the left Hand, and produc'd infinitely towards the Right, will be trac'd by any thing moving from the left to the right hand, in a finite equable Manner, or (ſince there is no Manner of Poſſibility of coming to an End of the ſaid ſtraight Line towards the right hand; for, by the Definition of the Word Infinity, it has none that Way) when two parallel ſtraight Lines would meet by continually producing them, or when an End would be found to the Circumference of a Circle by the continual Revolution of a Point thereon, or (which may ſeem more poſſible) when the exact Root of a ſurd Number would be found by a continual Extraction; that is to ſay, never.

So in affirming that an infinite Number of Finites produce Infinity, or that an infinite Number of Nothings produce ſome thing, thus much is meant, that any Number of Finites will not produce Infinity, or that any Number of Nothings will not produce any thing.

I

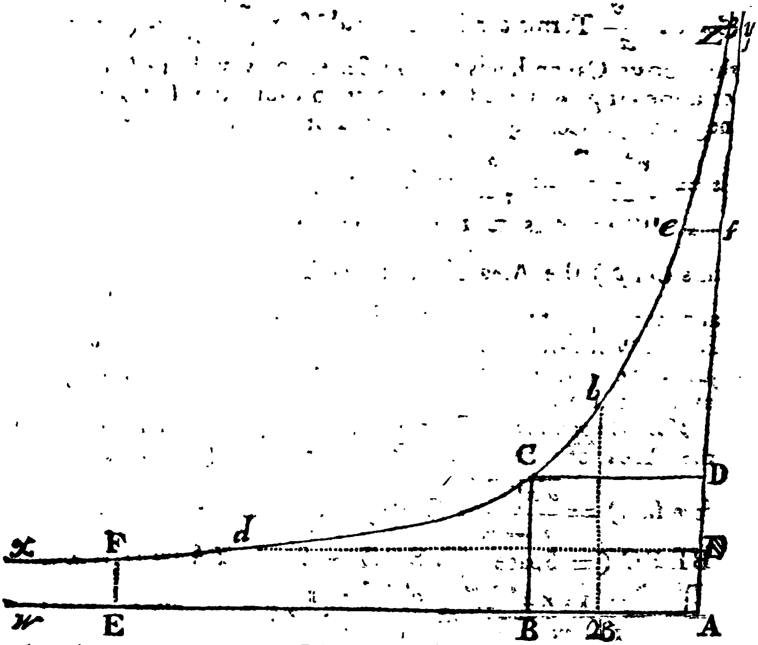
The

The CONCLUSION.

Having demonstrated, in *Part* XIII. as also in *Part* XIV. that, p being $=$ any affirmative whole Number, and $n =$ an indefinite Number, the Sum of 1^p, 2^p, 3^p, 4^p, 5^p, *&c.* to n Terms continued is $= \frac{n^{p+1}}{p+1} \multimap$, one may be apt thence to conclude that the same Canon should hold for that Series, although p were $=$ any Number whatsoever: But though such a Conclusion may seem very probable; yet, considering that Mathematicks ought to be founded on Certainty, I consider'd it further, in *Pages* 396, 397, 398, 399, and 400; as also in *Pages* 416 and 417, where I have demonstrated that the said Canon must likewise hold for the said Series when p is $=$ any affirmative fractional Number: So I proved that it holds in any Case wherein p is $=$ any affirmative Number whatsoever. It remain'd I should demonstrate that it would also hold in every Case wherein p is $=$ a negative Number. But here some Objections occurr'd to me, which, for some time, I thought insuperable, *viz.* when p is $= 0$, then the said Sum is $= \frac{n^{p+1}}{p+1}$ $= n$, and not equal to $\frac{n^{p+1}}{p+1} \multimap$; also when p is $= -1$, then the Sum of the said Series continued in Infinitum is, as found in the Theorem in *Page* 454 only $=$ *Neper*'s Logarithm of $\frac{1}{0}$; but, according to the above Canon, it is $= \frac{1}{0}$; and when p is $=$ any Number less than -1, as -2, -3, -4, $-\frac{3}{2}$, *&c.* then the said Sum, according to the above Canon, is negative.

Now, in order to account, in some measure, for these Objections, I shall have recourse to a Figure, to which any Series, with negative numerical Indices, are applicable

Let us therefore suppose the annexed one to represent any such Figure, in which the Lines A Y, A W, C Z and C X are suppos'd to be each of them infinite. And suppose $AB = x$,



BC ($\perp AB$ and $\parallel AY$) $= y$, and $y = x^{p} = x^{-m}$ (putting $-m = p$, since the Value of p must be negative in this Case.)

Then, according to our Method, suppose the Line A B to be divided into an indefinite Number of equal Parts, and each Part $= a$; and, at any proper Distance A 𝔅, which suppose $= za$ from A, draw 𝔅$b \parallel BC$ or AY: then, by the Property of the Figure, $\overline{za}|^{-m} =$ 𝔅b; therefore the Sum of the Areas of the circumscribing ▭s $= a \times : 0^{-m} + a^{-m} + \overline{2a}|^{-m} + \overline{3a}|^{-m} +$ &c. continued to $\frac{x}{a}$ Terms: is infinite, because of the first Term thereof, *viz.* $a \times 0^{-m}$. ☞ *See the Solution of* Prop. 4. But the Sum of the Areas of the indefinitely small ▭s inscribed in the Space YABCZ is $= a \times : a^{-m} + \overline{2a}|^{-m} + \overline{3a}|^{-m} + \overline{4a}|^{-m} +$ &c. to $\frac{x}{a}$ Terms

con-

continued : $= a^1 \times a^{-m} \times : 1^{-m} + 2^{-m} + 3^{-m} + 4^{-m} +$ &c. to $\frac{x}{a}$ Terms continued : $= a^{1-m} \times \frac{\overline{\frac{x}{a}}|^{1-m}}{1-m} \ominus$ (supposing the above Canon holds for this Series, only with the Difference of annexing $\ominus$ instead of $\oplus$ to it, because the Index here is negative) : Consequently the Area of the said Space YABCZ is $= \frac{x^{1-m}}{1-m} = \frac{xy}{1-m}$ (because y is equal to x^{-m}.)

1. When m is $\sqsupset 1$, then (by the *Corollary* to *Prop.* V. of this *Chap.*) the Area of the Space YABCZ is $= \frac{xy}{1-m}$; and consequently is finite. But the Area of its Complement BCXW is said to be more than infinite; but it may more justly be said to be infinite: And any Part of it may be measur'd thus,

Viz. Supposing AE $= bx$; then (EF being $\perp$ AE) the Area of the Space YAEFZ is (by what is before said) $= \frac{\overline{bx}|^{1-m}}{1-m}$: Consequently, the Area of the Space BEFC ($=$ Space YAEFZ $-$ Space YABCZ) is $= \frac{:\overline{b}|^{1-m} - 1 : \times x^{1-m}}{1-m} = \frac{b^{1-m} - 1}{1-m} xy$.

Again, the Area of the Space YDCZ (being $=$ Space YABCZ $- \square$ DABC) is $= \frac{xy}{1-m} - xy = \frac{x^{1-m}}{1-m} - x^{1-m} = \frac{mx^{1-m}}{1-m}$; wherefore, ef being $\perp$ AY, and putting $ef = \frac{x}{c}$, the Area of the Space Yefz is $= \frac{m \times \overline{\frac{x}{c}}|^{1-m}}{1-m} = \frac{mc^{m-1}x^{1-m}}{1-m}$.

Consequently, the Area of the Space fDCe is $= \frac{m - mc^{m-1}}{1-m} x^{1-m} = \frac{m - mc^{m-1}}{1-m} xy$.

Hence the Areas of the Spaces B E F C and Dfe C may be found in all Cases wherein m is $\sqsupset$ 1; and consequently, the Areas of the said Figures may be found in all Cases wherein m is $\sqsubset$ 1.

2. If m be $=$ 1, then the Area of the Space Y A B C Z is, according to the above Canon, $= \frac{xy}{0} = \frac{1}{0}$. Now, since by how much the greater any Number exceeding (say) 4 is, by so much the greater is the Disproportion between it and *Neper's* Logarithm thereof; and withal, if it be true that 1×0 is to 2×0, as 1 to 2, and, strictly speaking, not in any other Ratio; then there must be an immense Disproportion between $\frac{1}{0}$ and *Neper's* Logarithm of $\frac{1}{0}$, or of $\frac{1}{0}$; and, in that Case, the above Canon will not extend to the finding of the Area of the said Space.

But, if any Number of Nothings be to any other Number of Nothings in a Ratio of Equality, then $\frac{1}{0}$ may be said to be $=$ *Neper's* Logarithm of $\frac{1}{0}$; and the Consequence then is, that any infinite Number may, in this Case, express the Area of the aforesaid Space Y A B C Z.

3. When m is $\sqsubset$ 1, the Area, pursuant to the above Canon, of the Space of Y A B C Z is negative, or, (as it is commonly express'd) more than infinite. This may be partly accounted for thus, *viz.* I say the above Canon $\frac{xy}{1-m}$ with a contrary Sign, serves to find the Area of its Complement, *viz.* of the Space B C X W, which, in this Case, is finite, as may be thus prov'd:

Suppose A D $= y$ to be divided into an indefinite Number of equal Parts, and each Part $= a$; and, at any Distance za $=$ A D from A draw Dd || A W.

Then y being ($= \overline{DC}|^{-m}) = x^{-m}$; x is therefore $= y^{-\frac{1}{m}}$; and, of Consequence, Dd is $= \overline{za}|^{-\frac{1}{m}}$; wherefore (by the *Corollary* to *Prop.* V. since m is here suppos'd $\sqsubset$ 1, and, of consequence, $\frac{1}{m} \sqsupset 1$) the Area of the Space A D C X W $= a \times a^{-\frac{1}{m}} \times : 1^{-\frac{1}{m}} + 2^{-\frac{1}{m}} + 3^{-\frac{1}{m}} +$ &c. to $\frac{y}{a}$ Terms

con-

continued : $\bullet$ is $= a^{1-\frac{1}{m}} \times \frac{\left.\frac{y}{a}\right|^{1-\frac{1}{m}}}{1-\frac{1}{m}} = \frac{y^{1-\frac{1}{m}}}{1-\frac{1}{m}} = \frac{myx}{m-1}$;
from which subtracting xy (the Area of the ▭ ABCD) the Remainder is $\frac{xy}{m-1}$ = Area of the Space BCXW, which is manifestly $= -\frac{xy}{1-m}$. *W.W.D.*

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e	Roots	
0	$a=+10$	$a=+\sqrt[3]{+\frac{1}{2}d+\sqrt{+\frac{1}{4}}}$
0	$a=\begin{cases}+11\\+6\\-17\end{cases}$	$\begin{cases}1, 2\sqrt{\frac{1}{3}c}: Sine\, 90^d\\2, Sine\, 90^d : 2\sqrt{\frac{1}{3}}\end{cases}$